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Comparing Time Marching Schemes for Numerical Meteorology

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Abstract

The aim of this paper is to give a short comparison of explicit and semi-implicit time marching schemes used in numerical climate models. For this, simple model problems are investigated.
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AMS Classification. 65M06, 65M12

Zusammenfassung

Unser Ziel ist es, einen kurzen Vergleich von expliziten und semi-impliziten Zeitschrittverfahren, die in numerischen Klimamodellen benutzt werden, zu geben. Zu diesem Zweck werden einfache Modellprobleme untersucht.
Schlagworte. Numerische Meteorologie, Zeitschrittverfahren, partielle Differentialgleichungen

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1 Introduction

We give a short comparison of explicit and semi-implicit time marching schemes used in numerical climate models. For this as usual, simple model problems are investigated.

The paper collects in parts results from several papers comparing explicit and semi-implicit time marching schemes for numerical climate models concerning stability, amplification, phase errors and so on. Especially, we used the articles of Lilly [4], Young [5] and Durran [1].

2 Methods

An ordinary differential equation is given by

$$\frac{d\psi}{dt} = F(\psi).$$

We want to compare different time marching procedures all of them computing approximations $\phi^{(n)}$ to function values $\psi(n\Delta t)$. We denote $F^{(n)} = F(\phi^{(n)})$

1. Euler forward
$$\phi^{(n+1)} = \phi^{(n)} + (\Delta t)F^{(n)}$$

2. Euler's modified (trapezoidal rule)
$$\phi^{(n+1)} = \phi^{(n)} + \frac{\Delta t}{2}(F^{(n)} + F^{(n+1)})$$

3. Heun
$$\begin{aligned} \phi^{(n+1)*} &= \phi^{(n)} + (\Delta t)F^{(n)} \\ \phi^{(n+1)} &= \phi^{(n)} + \frac{\Delta t}{2}(F^{(n)} + F^{(n+1)*}) \end{aligned}$$

4. Classical Runge–Kutta

$$\begin{aligned}
\phi^{(n+1/2)*} &= \phi^{(n)} + \frac{\Delta t}{2} F^{(n)} \\
\overline{\phi}^{(n+1/2)} &= \phi^{(n)} + \frac{\Delta t}{2} F^{(n+1/2)*} \\
\phi^{(n+1)*} &= \phi^{(n)} + (\Delta t)\overline{F}^{(n+1/2)} \\
\phi^{(n+1)} &= \phi^{(n)} + \frac{\Delta t}{6}\left(F^{(n)} + 2F^{(n+1/2)*} + 2\overline{F}^{(n+1/2)} + F^{(n+1)*}\right)
\end{aligned}$$

Here, function values are approximated at $(n+1/2)\Delta t$ and $(n+1)\Delta t$ (the terms with a star or a bar) and a summed trapezoidal rule is applied.

5. Leapfrog (midpoint rule)

$$\phi^{(n+1)} = \phi^{(n-1)} + (2\Delta t)F^{(n)}$$

6. Adams-Bashforth 2nd order

$$\phi^{(n+1)} = \phi^{(n)} + \frac{\Delta t}{2}(3F^{(n)} - F^{(n-1)})$$

7. Adams-Bashforth 3rd order

$$\phi^{(n+1)} = \phi^{(n)} + \frac{\Delta t}{12}\left(23F^{(n)} - 16F^{(n-1)} + 5F^{(n-2)}\right)$$

The characteristics of these methods are given by the following table.

Method	Steps (Time Levels)	Order of accuracy	Number of RHS evaluations per step	Implicit or Explicit
1. Euler forward	1	1	1	E
2. Euler's modified	1	2	≥ 2	I
3. Heun	1	2	2	E
4. Classical Runge–Kutta	1	4	4	E
5. Leapfrog (midpoint rule)	2	2	1	E
6. Adams-Bashforth 2nd order	2	2	1	E
7. Adams-Bashforth 3rd order	3	3	1	E

3 Amplification and phase errors

Possible amplification and phase errors can for instance be seen by investigating the simple oscillation equation

$$\frac{d\psi}{dt} = i\omega\psi \tag{1}$$

with $\omega \in \mathbb{R}$. The analytic solution of (1) with initial condition $\psi(0) = \eta$ is given by

$$\psi(t) = \eta e^{i\omega t}. \tag{2}$$

The numerical solutions of (1) can be obtained by the finite-difference approximations given above such that

$$\phi^{(n)} = A^n \eta, \tag{3}$$

where A is the complex valued amplification factor. For the exact solution holds that

$$\psi(n\Delta t) = A_e^n \eta, \tag{4}$$

so the amplification factor for the exact solution is $A_e = e^{i\omega\Delta t}$, obviously $|A_e| = 1$. We denote

$$p = \omega\Delta t = \arg A_e.$$

The *relative amplitude error* in the numerical solution may be defined as

$$\frac{|A|}{|A_e|} = |A|.$$

The *relative phase error* is given as

$$R = \frac{\arg A}{\arg A_e} = \frac{1}{p} \arctan\left(\frac{\operatorname{Im} A}{\operatorname{Re} A}\right)$$

and usually will be given in terms of p.

In practice, the leap frog method is often used together with the Asselin filter in order to damp the computational mode and to get a better stability. We refer to that method as

5.a. Leapfrog with Asselin filter:

$$\begin{aligned} \phi^{(n+1)} &= \overline{\phi^{(n-1)}} + (2\Delta t) F^{(n)} \\ \overline{\phi^{(n)}} &= \phi^{(n)} + \frac{\nu}{2}\left(\overline{\phi^{(n-1)}} - 2\phi^{(n)} + \phi^{(n+1)}\right) \end{aligned}$$

Method	Amplification factor $\lvert A \rvert$	Frequency Error R
1. Euler forward	$(1+p^2)^{1/2}$	$\left(1-\frac{p^2}{3}\right)$
2. Euler's modified	1	$\left(1-\frac{p^2}{12}+\cdots\right)$
3. Heun	$(1+p^4)^{1/2}$	$\left(1+\frac{p^2}{6}+\cdots\right)$
4. Classical Runge–Kutta	$\left(1-\frac{p^6}{72}\right)^{1/2}$	$\left(1+\frac{p^4}{8}+\cdots\right)$
5. Leapfrog (midpoint rule)	1	$\left(1+\frac{p^2}{6}\right)$
5.a Leapfrog with Asselin	$\left(1-\frac{\nu p^2}{2(2-\nu)}-\cdots\right)$	$\left(1+\frac{(1+\nu)p^2}{3(2-\nu)}\right)$
6. Adams-Bashforth 2nd order	$\left(1+\frac{p^4}{4}+\cdots\right)$	$\left(1+\frac{5p^2}{12}+\cdots\right)$
7. Adams-Bashforth 3rd order	$\left(1-\frac{3p^4}{8}+\cdots\right)$	$\left(1+\frac{289p^4}{720}+\cdots\right)$

From this tabular, we can see that three methods (Euler forward, Heun and Adams-Bashforth 2nd order) have an amplification bigger than 1 and hence the numerical solution will get unstable already for this most simple test equation.

4 Amplification in the semi-implicit context

Many large meteorological models employ semi-implicit time differencing.

4.1 The leapfrog scheme combined with the trapezoidal rule

Mostly, the Asselin-filtered leapfrog scheme is used to represent the advection terms while the pressure gradient and the divergence terms are integrated using the trapezoidal method (Euler's modified). The basic properties may be illustrated by the following semi-implicit approximation to the oscillation equation (1):

$$\frac{\phi^{(n+1)} - \phi^{(n-1)}}{2\Delta t} = i\omega_1 \phi^{(n)} + \frac{i\omega_2}{2}(\phi^{(n+1)} + \phi^{(n-1)}), \tag{5}$$

where $\omega = \omega_1 + \omega_2$ and the Asselin filter is neglected. Kwizak and Robert [3] have shown that (5) will be stable whenever

$$\omega_1^2(\Delta t)^2 \leq 1 + \omega_2^2(\Delta t)^2. \tag{6}$$

Sufficient for (6) to be satisfied is $|\omega_1| \leq |\omega_2|$. In most atmospheric applications, ω_1 is identified with those frequencies characteristic to advection, ω_2 is identified with the frequencies characteristic of gravity wave propagation, and the condition $|\omega_1| \leq |\omega_2|$ is almost always satisfied. Since the accurate simulation of high-speed gravity waves is thought to be unnecessary in large-scale atmospheric models, the use of semi-implicit schemes allows to be efficiently integrated with a larger time step that would violate the CFL condition for gravity waves ($|\omega_2\Delta t| < t$) and lead to instability in a fully explicit scheme. We take a closer look to the amplification factor A of the scheme (5): Substituting

$$\phi^{(n+1)} = A^2\phi^{(n-1)}, \qquad \phi^{(n)} = A\phi^{(n-1)}$$

into (5) yields the equation

$$(1 - i\omega_2\Delta t)A^2 - 2i\omega_1\Delta t A - (1 + i\omega_2\Delta t) = 0. \tag{7}$$

We plotted the roots of (7) in the Figures 1 and 2 as a function of $\omega_1\Delta t$ for the values $\omega_2\Delta t = 0.75$, 2 and 4 and see that the behaviour even gets a bit better with some higher values of $\omega_2\Delta t$.

One might expect to obtain more accurate semi-implicit schemes by replacing the leapfrog part in (5) with a higher order formula as a third order Adams-Bashforth scheme or a fourth order Runge-Kutta formula.

4.2 The third order Adams-Bashforth scheme combined with the trapezoidal rule

We try the third order Adams-Bashforth scheme first and obtain

$$\frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} = \frac{i\omega_1}{12}(23\phi^{(n)} - 16\phi^{(n-1)} + 5\phi^{(n-2)}) + \frac{i\omega_2}{2}(\phi^{(n+1)} + \phi^{(n)}). \tag{8}$$

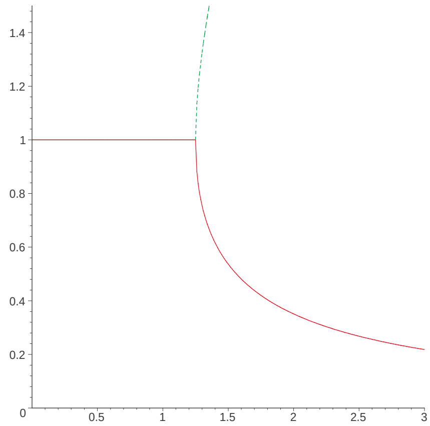


Figure 1: The leapfrog scheme combined with the trapezoidal rule: The graphs of the amplification factors as roots of (7) as a function of $\omega_1 \Delta t$ for the case $\omega_2 \Delta t = 0.75$.

This modification not only reduces the truncation error in the ω_1 component of the solution but also improves the accuracy of the ω_2 component by halving the step size of the trapezoidal integration. Unfortunately, the new semi-implicit scheme does not herit the robust stability of the original semi-implicit scheme. The amplification factor for (8) satisfies

$$\left(1 - \frac{i\omega_2 \Delta t}{2}\right) A^3 - \left(1 + \frac{23 i\omega_1 \Delta t}{12} + \frac{i\omega_2 \Delta t}{2}\right) A^2 + \frac{4 i\omega_1 \Delta t}{3} A - \frac{5 i\omega_1 \Delta t}{12} = 0. \quad (9)$$

The magnitude of the three roots of (9) is plotted in Figure 3 as a function of $\omega_1 \Delta t$ for the case $\omega_2 \Delta t = 2$ (right-hand side). Note that the physical mode is always unstable. Stable integrations can be achieved by reducing $\omega_2 \Delta t$, but there is little point in using a semi-implicit scheme unless the high-frequency modes can be stably integrated at Courant numbers ($\omega_2 \Delta t$) much greater than unity.

4.3 The third order Adams-Bashforth scheme combined with the third order Adams-Moulton scheme

The question arises whether the combination of the Adams-Bashforth scheme with the trapezoidal rule is a natural one. The most natural choice for an

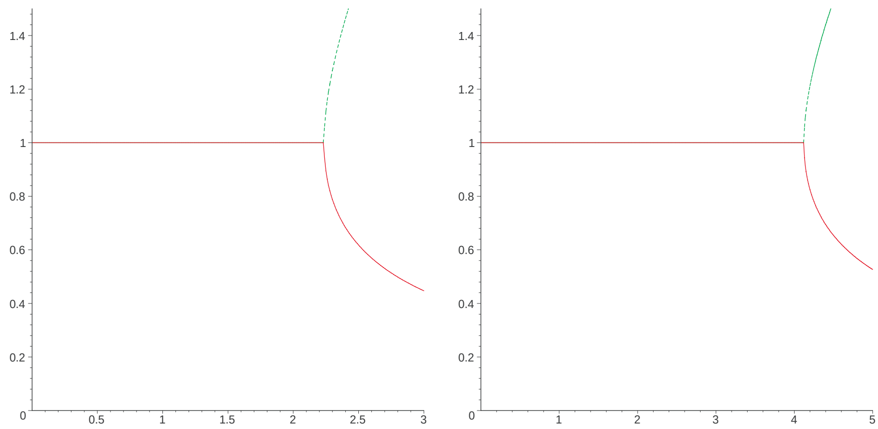


Figure 2: The leapfrog scheme combined with the trapezoidal rule: The graphs of the amplification factors as roots of (7) as a function of $\omega_1\Delta t$ for the case $\omega_2\Delta t = 2$ (left) and $\omega_2\Delta t = 4$ (right).

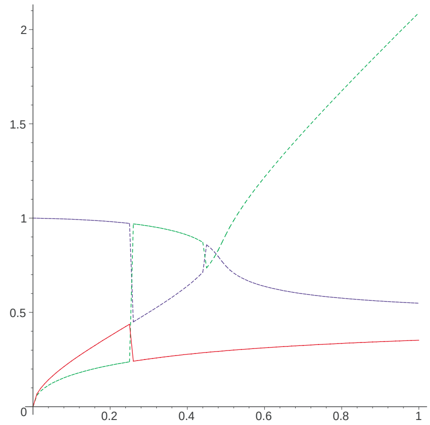
implicit scheme to be combined with an Adams-Bashforth formula seems to be the Adams-Moulton scheme of the same (third) order, what gives

$$\begin{aligned}\frac{\phi^{(n+1)} - \phi^{(n)}}{\Delta t} &= \frac{i\omega_1}{12}(23\phi^{(n)} - 16\phi^{(n-1)} + 5\phi^{(n-2)}) \\ &+ \frac{i\omega_2}{24}(9\phi^{(n+1)} + 19\phi^{(n)} - 5\phi^{(n-1)} + \phi^{(n-2)})\end{aligned} \tag{10}$$

From this, we obtain an equation for the amplification factor for (10)

$$\begin{aligned}\Big(1 - \frac{3i\omega_2\Delta t}{8}\Big)A^3 - \Big(1 + \frac{23i\omega_1\Delta t}{12} + \frac{19i\omega_2\Delta t}{24}\Big)A^2 \\ + \Big(\frac{4i\omega_1\Delta t}{3} + \frac{5i\omega_2\Delta t}{24}\Big)A - \Big(\frac{5i\omega_1\Delta t}{12} + \frac{i\omega_2\Delta t}{24}\Big) &= 0.\end{aligned} \tag{11}$$

We again plot the magnitude of the three roots of (11) (Figure 4, right) as a function of $\omega_1\Delta t$ for the case $\omega_2\Delta t = 2$. In this case, the amplification of the physical mode is even worse. Here, even reducing $|\omega_2\Delta t|$ does not really help much as can be seen from the left-hand side of Figure 4.



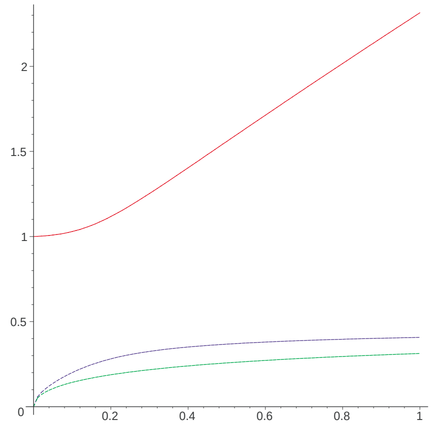


Figure 3: The third order Adams-Bashforth scheme combined with the trapezoidal rule: The graphs of the amplification factors as the three roots of (9) as a function of $\omega_1 \Delta t$ for the case $\omega_2 \Delta t = 0.75$ (left) and $\omega_2 \Delta t = 2$ (right).

4.4 The fourth order Runge-Kutta scheme combined with the trapezoidal rule

In case of the classical Runge-Kutta scheme, the semi-implicit variant using the trapezoidal rule may look like

$$\begin{aligned} \phi^{(n+1/2)*} &= \phi^{(n)} + \frac{i\omega_1 \Delta t}{2}\phi^{(n)} \\ \overline{\phi}^{(n+1/2)} &= \phi^{(n)} + \frac{i\omega_1 \Delta t}{2}\phi^{(n+1/2)*} \\ \phi^{(n+1)*} &= \phi^{(n)} + (i\omega_1 \Delta t)\overline{\phi}^{(n+1/2)} \\ \phi^{(n+1)} &= \phi^{(n)} + \frac{i\omega_1 \Delta t}{6}\big(\phi^{(n)} + 2\phi^{(n+1/2)*} + 2\overline{\phi}^{(n+1/2)} + \phi^{(n+1)*}\big) \\ &\quad + \frac{i\omega_2 \Delta t}{2}(\phi^{(n+1)} + \phi^{(n)}). \end{aligned} \tag{12}$$

From this, we obtain the amplification factor as

$$\begin{aligned} A &= \frac{1}{1-\frac{i\omega_2\Delta t}{2}}\Big(\frac{i\omega_1 \Delta t}{2} + 1 + \frac{i\omega_1 \Delta t}{6}\Big(6 + i\omega_1 \Delta t \\ &\quad + i\omega_1 \Delta t\big(1 + \frac{i\omega_1 \Delta t}{2}\big) + i\omega_1 \Delta t\big(1 + \frac{i\omega_1 \Delta t}{2}\big(1 + \frac{i\omega_1 \Delta t}{2}\big)\big)\Big)\Big). \end{aligned} \tag{13}$$

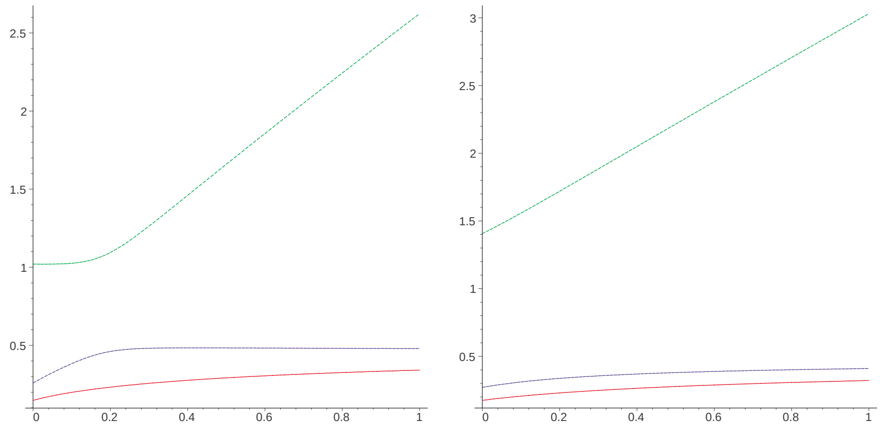


Figure 4: The third order Adams-Bashforth scheme combined with the third order Adams-Moulton scheme: The graphs of the amplification factors as the three roots of (11) as a function of $\omega_1\Delta t$ for the case $\omega_2\Delta t = 0.75$ (left) and $\omega_2\Delta t = 2$ (right).

We plotted the amplification factor in Figure 5 as a function of $\omega_1\Delta t$ for the cases $\omega_2\Delta t = 0$ (left) and $\omega_2\Delta t = 0.75$ (right) and in Figure 6 for the cases $\omega_2\Delta t = 2$ (left) and $\omega_2\Delta t = 4$ (right). The curves look almost the same for higher values of $\omega_2\Delta t$, too. So, we always find regions in terms of $\omega_1\Delta t$ where the amplification factor is close to one or less.

5 Stability in case of the advection equation

One of the most simple time-dependent partial differential equations is the advection equation

$$\frac{\partial\psi}{\partial t} + c\frac{\partial\psi}{\partial x} = 0. \tag{14}$$

As an initial condition we assume a simple wave

$$\psi(x,0) = ae^{i\omega x}. \tag{15}$$

The equation (14) can be solved by the method of separation of variables by letting

$$\psi(x,t) = g(t)f(x).$$

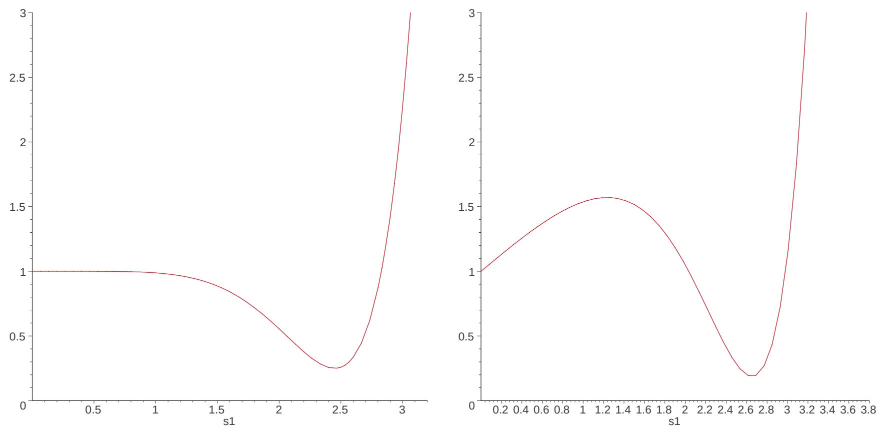


Figure 5: The fourth order Runge-Kutta scheme combined with the trapezoidal rule: The graph of the amplification factor (13) as a function of $\omega_1 \Delta t$ for the case $\omega_2 \Delta t = 0$ (left) and $\omega_2 \Delta t = 0.75$ (right).

Substituting this in (14) leads to

$$g'(t)f(x) + cg(t)f'(x) = 0$$

and hence,

$$\frac{g'(t)}{g(t)} = -c\frac{f'(x)}{f(x)} = \text{const.} =: -k.$$

Integration gives

$$g(t) = a_1 e^{-kt}, \qquad f(x) = a_2 e^{-kx/c}.$$

Together with the initial condition (15), we have

$$\psi(x,t) = ae^{i\omega(x-ct)}$$

The stability of the numerical solution does not only depend on the time marching scheme but also on space discretization. For all schemes we use equidistant time and space discretization:

$$\begin{aligned} x_m &= m\Delta x, \quad m = 0, \pm 1, \pm 2, \ldots \\ t_n &= n\Delta t, \quad n = 0, 1, 2, \ldots \end{aligned}$$

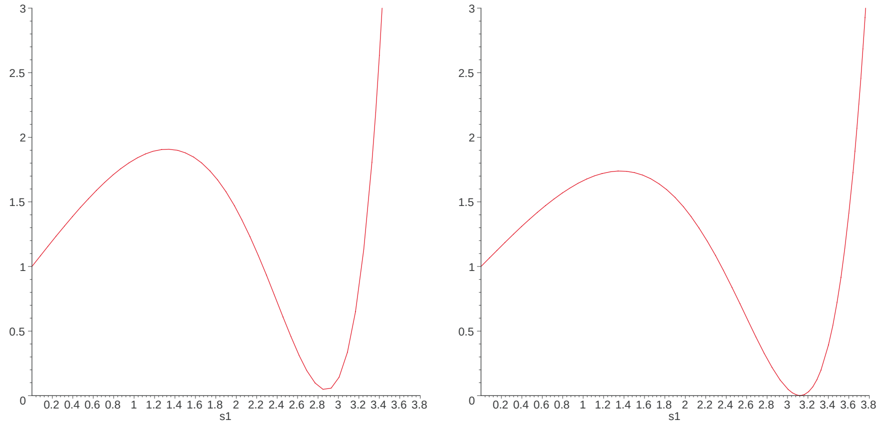


Figure 6: The fourth order Runge-Kutta scheme combined with the trapezoidal rule: The graph of the amplification factor (13) as a function of $\omega_1 \Delta t$ for the case $\omega_2 \Delta t = 2$ (left) and $\omega_2 \Delta t = 4$ (right)

with the approximation

$$\phi^{(m,n)} = \psi(x_m, t_n)$$

of the function values and a centered space dicretization, i.e.,

$$F^{(m,n)} = -\frac{c}{2\Delta x}(\phi^{(m+1,n)} - \phi^{(m-1,n)}).$$

5.1 The Euler method

For the advection equation the Euler methods reads as

$$\phi^{(m,n+1)} = \phi^{(m,n)} - \frac{c\Delta t}{2\Delta x}(\phi^{(m+1,n)} - \phi^{(m-1,n)}).$$

Assuming a solution

$$\phi^{(m,n)} = A_n e^{i\omega m \Delta x}, \tag{16}$$

we compute

$$\begin{aligned} A_{n+1} e^{i\omega m \Delta x} &= A_n e^{i\omega m \Delta x} - \frac{c\Delta t}{2\Delta x} A_n (e^{i\omega(m+1)\Delta x} - e^{i\omega(m-1)\Delta x}) \\ A_{n+1} &= A_n \left(1 - \frac{c\Delta t}{2\Delta x}(e^{i\omega\Delta x} - e^{-i\omega\Delta x})\right) \\ A_{n+1} &= A_n \left(1 - i\frac{c\Delta t}{\Delta x}\sin\omega\Delta x\right). \end{aligned}$$

We denote

$$\sigma = \frac{c\Delta t}{\Delta x} \sin \omega \Delta x,$$

$\theta = \arctan \sigma$ and have

$$A_{n+1} = A_n \left(1 - i\sigma\right) = A_n \left(1 + \sigma^2\right)^{1/2} e^{-i\theta}.$$

With an initial condition $\psi(x,0) = ae^{i\omega x}$ this would yield a numerical solution

$$\phi^{(m,n)} = a \left(1 + \sin^2 \omega \Delta x\right)^{n/2} e^{i\omega(m\Delta x - n\theta/\omega)}$$

which shows exponential growth for almost all wavelengths. This means that the Euler method with centered space discretization is never stable for the advection equation.

For further use, we again note the relation

$$\phi^{(m+1,n)} - \phi^{(m-1,n)} = (2i \sin \omega \Delta x)\phi^{(m,n)} \tag{17}$$

5.2 Euler's modified Method

This method is the only implicit method we discuss here. For the advection equation it is given by

$$\phi^{(m,n+1)} = \phi^{(m,n)} - \frac{c\Delta t}{4\Delta x}\left(\left(\phi^{(m+1,n+1)} - \phi^{(m-1,n+1)}\right) + \left(\phi^{(m+1,n)} - \phi^{(m-1,n)}\right)\right),$$

i.e.,

$$\phi^{(m,n+1)} = \phi^{(m,n)} - \frac{ic\Delta t}{2\Delta x} \sin \omega \Delta x (\phi^{(m,n+1)} + \phi^{(m,n)}).$$

This gives

$$\phi^{(m,n+1)} = \phi^{(m,n)} \frac{1 - \sigma i/2}{1 + \sigma i/2} = \lambda \phi^{(m,n)}$$

with $|\lambda| \equiv 1$ being a quotient of conjugate complex numbers. So, Euler's modified method is unconditionally stable for the advection equation with centered space discretization.

5.3 Heun's Method

Heun's method is somewhere in between the explicit and the implicit Euler method discussed above, because it uses one explicit Euler step to approximate the input of an implicit Euler step:

$$\begin{aligned}
\phi^{(m,n+1)*} &= \phi^{(m,n)} - \frac{c\Delta t}{2\Delta x}\big(\phi^{(m+1,n)} - \phi^{(m-1,n)}\big),\\
\phi^{(m,n+1)} &= \phi^{(m,n)} - \frac{c\Delta t}{4\Delta x}\big((\phi^{(m+1,n+1)*} - \phi^{(m-1,n+1)*})\\
&\quad + (\phi^{(m+1,n)} - \phi^{(m-1,n)})\big).
\end{aligned}$$

We again assume a solution (16) resulting from a single wave and obtain

$$\begin{aligned}
\phi^{(m,n+1)*} &= (1-\sigma i)\phi^{(m,n)},\\
\phi^{(m,n+1)} &= \phi^{(m,n)} - \frac{1}{2}\sigma i\phi^{(m,n)} - \frac{1}{2}\sigma i(1-\sigma i)\phi^{(m,n)},\\
&= \phi^{(m,n)}\Big(1 - \frac{1}{2}\sigma^2 - \sigma i\Big).
\end{aligned}$$

This scheme would be stable for such σ satisfying

$$\left|1 - \frac{1}{2}\sigma^2 - \sigma i\right| \leq 1.$$

But

$$\left|1 - \frac{1}{2}\sigma^2 - \sigma i\right|^2 = 1 + \frac{1}{4}\sigma^4$$

what is almost always bigger than 1. So, Heun's method will be never stable for the advection equation with centered space differencing.

5.4 Classical Runge-Kutta Method

The classical Rung-Kutta method for the advection equation is

$$
\begin{aligned}
\phi^{(m,n+1/2)*} &= \phi^{(n)} - \frac{c\Delta t}{4\Delta x}\big(\phi^{(m+1,n)} - \phi^{(m-1,n)}\big)\\
\overline{\phi}^{(m,n+1/2)} &= \phi^{(n)} - \frac{c\Delta t}{4\Delta x}\big(\phi^{(m+1,n+1/2)*} - \phi^{(m-1,n+1/2)*}\big)\\
\phi^{(m,n+1)*} &= \phi^{(n)} + \frac{c\Delta t}{2\Delta x}\big(\overline{\phi}^{(m+1,n+1/2)} - \overline{\phi}^{(m-1,n+1/2)}\big)\\
\phi^{(m,n+1)} &= \phi^{(n)} + \frac{\Delta t}{12\Delta x}\Big(\big(\phi^{(m+1,n)} - \phi^{(m-1,n)}\big) + 2\big(\phi^{(m+1,n+1/2)*}\\
&\quad - \phi^{(m-1,n+1/2)*}\big) + 2\big(\overline{\phi}^{(m+1,n+1/2)} - \overline{\phi}^{(m-1,n+1/2)}\big)\\
&\quad + \big(\phi^{(m+1,n+1)*} - \phi^{(m-1,n+1)*}\big)\Big)
\end{aligned}
$$

For a solution of form (16) this means that

$$
\begin{aligned}
\phi^{(m,n+1)} &= \phi^{(m,n)}\Big(1 - \frac{1}{6}i\sigma\big(6 - i\sigma - i\sigma\big(1 - \frac{1}{2}i\sigma\big)\\
&\quad - i\sigma\big(1 - \frac{1}{2}i\sigma\big(1 - \frac{1}{2}i\sigma\big)\big)\big)\Big)\\
&= \lambda\phi^{(m,n)}.
\end{aligned}
$$

This raises the question which σ satisfies the stability conditon $|\lambda| \le 1$.

From the plot of $|\lambda|^2$ in Figure 7, we find that this is the case for all σ with

$$|\sigma| = \left|\frac{c\Delta t}{\Delta x}\right| |\sin\omega\Delta x| \le 2.828427.$$

This should hold for all wavelengths, what gives the criterion

$$\left|\frac{c\Delta t}{\Delta x}\right| \le 2.828427.$$

5.5 The Leapfrog Scheme

The leap frog scheme is widely used in computational meteorology. It is a two-level scheme, because it uses not only values from n-th time step, but also for time step $n-1$ to computed values for time $n+1$:

$$\phi^{(m,n+1)} = \phi^{(m,n-1)} - \frac{c\Delta t}{\Delta x}\big(\phi^{(m+1,n)} - \phi^{(m-1,n)}\big).$$

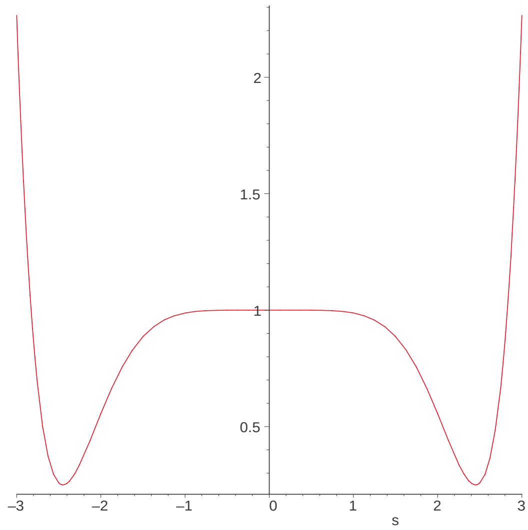


Figure 7: The graph of $|\lambda|^2$ for the amplification factor of the classical Runge-Kutta scheme.

For doing the analysis, we introduce a new variable

$$\chi^{(m,n)} = \phi^{(m,n-1)}$$

because of this two-level structure. This gives

$$\begin{aligned} \phi^{(m,n+1)} &= \chi^{(m,n)} - \frac{c\Delta t}{\Delta x}(\phi^{(m+1,n)} - \phi^{(m-1,n)}), \\ \chi^{(m,n+1)} &= \phi^{(m,n)}. \end{aligned}$$

Assume form (16) for $\phi^{(m,n)}$ and additionally

$$\chi^{(m,n)} = B_n e^{i\omega m\Delta x}. \tag{18}$$

We substitute it in the scheme and get

$$\begin{aligned} A_{n+1} &= B_n - 2i\sigma A_n \\ B_{n+1} &= A_n. \end{aligned}$$

This reads in matrix form as

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} -2i\sigma & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

We determine the eigenvalues of the matrix involved as the zeros of the characteric polynomial

$$\begin{vmatrix} -2i\sigma - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 2i\sigma\lambda - 1 = 0$$

and obtain

$$\lambda_{1/2} = -i\sigma \pm \sqrt{1-\sigma^2},$$

so, for $|\sigma| \leq 1$,

$$|\lambda_{1/2}|^2 = \sigma^2 + 1 - \sigma^2 = 1$$

in which case both modes are neutral. For $|\sigma| > 1$:

$$\lambda_{1/2} = i(-\sigma \pm \sqrt{\sigma^2 - 1}),$$

so at least $|\lambda_2| > 1$ and the scheme will become unstable.

We got $|\sigma| \leq 1$ as a necessary condition for the scheme to be stable, i.e.

$$|\sigma| = \left|\frac{c\Delta t}{\Delta x}\right| \, |\sin \omega \Delta x| \leq 1.$$

This should hold for all wavelengths, hence the condition becomes the well-known CFL-condition

$$\left|\frac{c\Delta t}{\Delta x}\right| \leq 1.$$

5.6 Adams-Bashforth Method of Second Order

For the advection equation, the two level Adams-Bashforth method of second order can be written as

$$\phi^{(m,n+1)} = \phi^{(m,n)} - \frac{c\Delta t}{4\Delta x}\big(3(\phi^{(m+1,n)} - \phi^{(m-1,n)}) - (\phi^{(m+1,n-1)} - \phi^{(m-1,n-1)})\big).$$

As before, we indroduce $\chi^{(m,n)} = \phi^{(m,n-1)}$ and have

$$\begin{aligned} \phi^{(m,n+1)} &= \phi^{(m,n)} - \frac{c\Delta t}{4\Delta x}\big(3(\phi^{(m+1,n)} - \phi^{(m-1,n)}) - (\chi^{(m+1,n)} - \chi^{(m-1,n)})\big) \\ \chi^{(m,n+1)} &= \phi^{(m,n)}. \end{aligned}$$

Assuming (16) and (18) gives

$$\begin{aligned} A_{n+1} &= A_n - \frac{3}{2} i\sigma A_n + \frac{1}{2} i\sigma B_n \\ B_{n+1} &= A_n, \end{aligned}$$

what is in matrix form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{3}{2} i\sigma & \frac{1}{2} i\sigma \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

The eigenvalues can be computed from

$$\begin{vmatrix} 1 - \frac{3}{2} i\sigma - \lambda & \frac{1}{2} i\sigma \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \left(\frac{3}{2} i\sigma - 1\right)\lambda - \frac{1}{2} i\sigma = 0$$

The roots are

$$\lambda_{1/2} = \frac{1}{4} i \left(-2i - 3\sigma \pm \sqrt{9\sigma^2 + 4i\sigma - 4} \right).$$

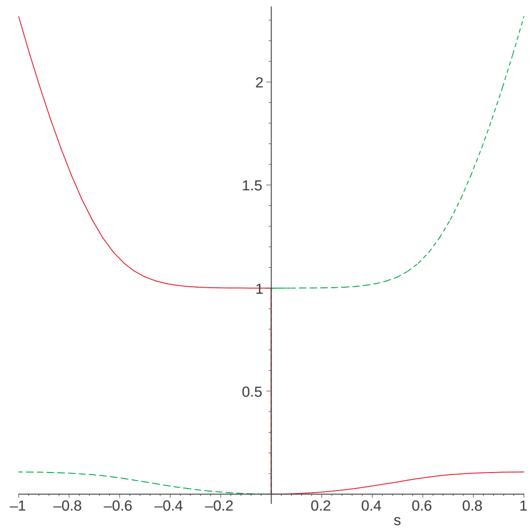


Figure 8: The graphs of $|\lambda_{1/2}|^2$ for the eigenvalues of the amplification matrix of the Adams-Bashforth scheme of second order.

The question now is whether there are σ for which $|\lambda_{1/2}| \leq 1$ ensures stability of the numerical solution. A look at the plots of $|\lambda_{1/2}|^2$ in Figure 8 shows that this is never the case.

5.7 Adams-Bashforth Method of Third Order

The three level Adams-Bashforth method of third order has the form

$$\begin{aligned}\phi^{(m,n+1)} &= \phi^{(m,n)} - \frac{c\Delta t}{24\Delta x}\big(23(\phi^{(m+1,n)} - \phi^{(m-1,n)}) - 16(\phi^{(m+1,n-1)} \\ &\quad - \phi^{(m-1,n-1)}) + 5(\phi^{(m+1,n-2)} - \phi^{(m-1,n-2)})\big)\end{aligned}$$

for the advection equation, We denote $\chi^{(m,n)} = \phi^{(m,n-1)}$, $\xi^{(m,n)} = \phi^{(m,n-2)}$ and obtain

$$\begin{aligned}\phi^{(m,n+1)} &= \phi^{(m,n)} - \frac{c\Delta t}{24\Delta x}\big(23(\phi^{(m+1,n)} - \phi^{(m-1,n)}) - 16(\chi^{(m+1,n)} \\ &\quad - \chi^{(m-1,n)}) + 5(\xi^{(m+1,n)} - \xi^{(m-1,n)})\big) \\ \chi^{(m,n+1)} &= \phi^{(m,n)} \\ \xi^{(m,n+1)} &= \chi^{(m,n)}\end{aligned}$$

We assume again (16) and (18) and additionally $\xi^{(m,n)} = C_n e^{i\omega m\Delta x}$. This yields

$$\begin{aligned}A_{n+1} &= A_n - \frac{i\sigma}{12}(23A_n - 12B_n + 5C_n) \\ B_{n+1} &= A_n, \\ C_{n+1} &= B_n,\end{aligned}$$

in matrix form

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \\ C_{n+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{23}{12}i\sigma & \frac{4}{3}i\sigma & -\frac{5}{12}i\sigma \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix}.$$

The eigenvalues satisfy

$$\begin{vmatrix} 1 - \frac{23}{12}i\sigma - \lambda & \frac{4}{3}i\sigma & -\frac{5}{12}i\sigma \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = \lambda^2\Big(1 - \frac{23}{12}i\sigma - \lambda\Big) - \frac{5}{12}i\sigma + \frac{4}{3}i\sigma\lambda = 0$$

The expesions for the roots are rather lengthy so we skip them. The plots of $|\lambda_{1/2/3}|^2$ in Figure 9 show their dependency on σ. We can see, that there are regions of σ for which $|\lambda_{1/2/3}| \le 1$.

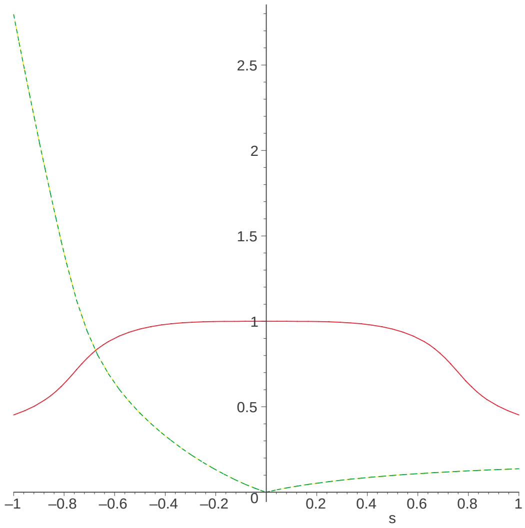


Figure 9: The graphs of $|\lambda_{1/2/3}|^2$ for the eigenvalues of the amplification matrix of the Adams-Bashforth scheme of third order.

A closer look shows that

$$|\sigma| \leq 0.723535$$

ensures of the stability criterion $|\lambda_{1/2/3}| \leq 1$ to be satisfied. The same arguments as for the leapfrog scheme lead to the condition

$$\left| \frac{c\Delta t}{\Delta x} \right| \leq 0.723535.$$

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