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The second volume moment of the typical cell and higher moments of edge lengths of the spatial Poisson-Voronoi tessellation

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Abstract This paper presents analytical results for higher moments of characteristics of a Voronoi tessellation generated by a homogeneous Poisson point process in the three-dimensional Euclidean space. The second moment of the volume of the typical cell as well as higher moments for the edge length distribution and the linear contact distribution are given. These characteristics are calculated analytically and presented in a unified form.

Keywords Poisson-Voronoi tessellation · Moments · Zero cell · Typical cell · Edge length · Linear contact distribution

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1 Introduction

The Voronoi tessellation of a point process is a popular stochastic geometrical model applied in many fields of science and engineering, see e. g. [13]. A commonly used special case is that of complete randomness in which the generating point pattern is a homogeneous Poisson point process. This so-called Poisson-Voronoi tessellation has been studied by many researchers, both analytically and by means of simulation.

In view of such a widespread use of this model, there is great interest in obtaining knowledge of geometrical characteristics. In particular, this concerns distributional properties of cell volumes and also edge lengths or characteristics of related linear and planar sections.

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Moments of geometrical characteristics like cell volumes and edge lengths are important aims for a quantitative evaluation of random structures. Meijering [6] derived the mean values of several cell characteristics of the planar and spatial Poisson-Voronoi tessellation. These formulae have been generalized by Miles [7] and Møller [9, Sect. 7] to higher dimensions Hayen and Quine [5] calculated the second and third moment of the cell area of the planar Poisson-Voronoi tessellation in double and five-fold integral form, respectively. Gilbert [4] provided a double integral formula for the volume of the zero cell (in close relationship to the second volume moment of the typical cell) of the Poisson-Voronoi tessellation in arbitrary dimensions including numerical results for the planar and spatial case.

However, further formulae for higher moments of the cell volume are unknown. To compense this lack, other quantities like edge lengths have been studied. Often they are in direct relationship to material properties and therefore of great interest for many applicants. Many results concerning the Poisson-Voronoi tessellation are summarized in [13, Sect. 5.5.1].

Brakke [1,2] calculated second moments of several characteristics and correlations between them for the planar and spatial Poisson-Voronoi tessellation and the plane cross section. Furthermore, double integral formulae for the moments of the edge length distribution including cross sections and the linear contact distribution in arbitrary dimension are given by Muche [11,12].

Whilst most of the above mean values are given analytically, for the other characteristics only numerical results based on integral representations or simulation are available [13], and their analytical structure is still unknown. The aim of the paper is to give some analytical results for the important special case of the three-dimensional Poisson-Voronoi tessellation. Results are given for the second moment of the volume of the typical cell, for the moments of the edge length distribution and the linear contact distribution from second to twelfth order (namely these cases, where the order of the moment k and the sectional dimension s fulfil the condition $k+2s \equiv 0 (\mathrm{mod}\ 3)$. These higher moments of the edge lengths are given in the following unified form

$$\mathrm{E}L_s^k = \left(\frac{3}{\pi\lambda}\right)^{\frac{k}{3}} \frac{4^{1-\frac{s}{3}}}{\Gamma\left(\frac{2s}{3}\right)} \left(a_{k,s}\left(16\mathscr{A} - 3\pi^2\right) + b_{k,s}\sqrt{3}\,\pi + c_{k,s}\right),$$

where $a_{k,s}, b_{k,s}$ and $c_{k,s}$ are rational coefficients, and $\mathscr{A}$ is a real constant occuring in all considered cases. The constant $\mathscr{A}$ can be given as a sum of analytical standard functions (dilogarithms) with complex arguments, or in form of a sum of two real integrals. As a consequence, simple analytical relationships between these characteristics can be derived by comparison of coefficients.

The paper is organized as follows: Section 2 introduces notation and gives some previous results. Based on these, in Section 3 the new results (all numbered formulae) are presented. Finally, Section 4 gives the proofs of them.

2 Fundamentals

Let Φ be a homogeneous Poisson point process of intensity λ in the three-dimensional Euclidean space $\mathbb{R}^3$. Denote by $\mathscr{V}_3$ the corresponding Voronoi tessellation, defined as the system $\{C(x,\Phi) : x \in \Phi\}$ of all cells

$$C(x,\Phi) = \{z \in \mathbb{R}^3 : \|z-x\| \le \|z-y\|, \forall y \in \Phi \setminus \{x\}\}, \quad x \in \Phi.$$

Denote by C_0 the *zero cell* of $\mathscr{V}_3$, i. e. the (almost surely unique) cell containing the origin o, and by C the *typical cell* of $\mathscr{V}_3$, where 'typical' is used in the sense of [14, p. 117].

The intersection of $\mathscr{V}_3$ by a plane or a line is a (lower-dimensional) tessellation as well. Denote by $\mathscr{V}_2$ the intersection of $\mathscr{V}_3$ with the (x_1,x_2)-plane (planar section), and by $\mathscr{V}_1$ the intersection of $\mathscr{V}_3$ with the x_1-axis (linear section). Since the Poisson-Voronoi tessellation $\mathscr{V}_3$ is isotropic, distributional properties of both the planar and the linear section do not depend on this specific choice for the intersecting plane and line, respectively.

Note that for a spatial Poisson-Voronoi tessellation both the linear section $\mathscr{V}_1$ and the planar section $\mathscr{V}_2$ itself is by no means the Voronoi tessellation of any generating point process in $\mathbb{R}^1$ and $\mathbb{R}^2$, respectively. A corresponding assertion for the planar section $\mathscr{V}_2$ has been a long-standing question in stochastic geometry. Chiu, van de Weygaert and Stoyan [3] showed that the intersection between an arbitrary but fixed plane and the spatial Poisson-Voronoi tessellation is not a planar Voronoi tessellation. For the linear section $\mathscr{V}_1$ the reasoning is as follows. If $\mathscr{V}_1$ would be a linear Voronoi tessellation, every two consecutive points of its generating point process must mutually be reflections at the intersection points of the chords. Consider three consecutive chords in $\mathscr{V}_1$ with lengths denoted by l_1, l_2 and l_3, respectively. Since Φ is a homogeneous Poisson point process there is a positive probability that this reflection property is not satisfiable, e. g. if the inequality $l_1 + l_3 < l_2$ holds.

The random length of the typical edge of $\mathscr{V}_s$ is denoted by L_s, $s = 1,2,3$. Its moments are given by

$$\begin{aligned}\mathrm{E}L_s^k = {} & \frac{3^{2+\frac{k}{3}}4^{\frac{2s}{3}+\frac{1}{2}}\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(\frac{k+2s}{3}+2\right)}{\pi^{\frac{k}{3}+\frac{1}{2}}\Gamma\left(\frac{2s}{3}\right)\Gamma(s+2)\lambda^{\frac{k}{3}}} \\ & \times \int_0^{\pi}\int_0^{\pi-\alpha} \frac{(\sin\alpha\sin\beta)^{2s+2}\sin(\alpha+\beta)^k(1+\cos\alpha)^2(1+\cos\beta)^2\mathrm{d}\beta\mathrm{d}\alpha}{\left(\sin^3\alpha(2+\cos\beta(2+\sin^2\beta))+\sin^3\beta(2+\cos\alpha(2+\sin^2\alpha))\right)^{2+\frac{k+2s}{3}}},\end{aligned} \quad (1)$$

see [11, Eqn. (20)] for the special case $d = 3$. Here, $\Gamma(t)$ denotes Euler's Γ-function

$$\Gamma(t) = \int_0^{\infty} \tau^{t-1}\exp\{-\tau\}\,\mathrm{d}\tau.$$

A further characteristic of interest is the *linear contact distribution function* $H_l(r)$, $r \ge 0$, of the random closed set formed by the union of all cell boundaries of the tessellation $\mathscr{V}_3$. In this case, $H_l(r)$ is the probability that the distance R of the origin o to the boundary of C_0 along any fixed direction is not greater than r, i. e. H_l is the cumulative distribution function of the random variable R. Due to isotropy, the values of H_l do not depend on the specific choice of the direction. For definition and properties see [14, p. 71ff].

In [12, Eqn. (20)] Muche provides the following formula for the moments $\mathrm{E}R^k$, $k = 1,2,\ldots$ of $H_l(r)$:

$$\mathrm{E}R^k = \frac{2\cdot 3^{1+\frac{k}{3}}\Gamma\left(\frac{k}{3}+2\right)}{\pi^{\frac{k}{3}}\lambda^{\frac{k}{3}}} \times \int_0^{\pi}\int_0^{\pi-\alpha} \frac{(\sin^2\alpha\sin\beta)^2\sin(\alpha+\beta)^k(1+\cos\beta)^2\mathrm{d}\beta\mathrm{d}\alpha}{\left(\sin^3\alpha(2+\cos\beta(2+\sin^2\beta))+\sin^3\beta(2+\cos\alpha(2+\sin^2\alpha))\right)^{2+\frac{k}{3}}}. \tag{2}$$

Recall that there is a close relationship between the linear contact distribution function $H_l(r)$ and the edge length distribution function $F_L(r)$ in the linear section $\mathscr{V}_1$, the latter commonly being called *chord length distribution function*. This relationship is given by

$$H_l(r) = \frac{1}{\mathrm{E}L_1}\int_0^r (1-F_L(\tau))\mathrm{d}\tau, \tag{3}$$

see [14, p. 208], with mean chord length

$$\mathrm{E}L_1 = \left(\frac{3}{4\pi\lambda}\right)^{\frac{1}{3}}\frac{1}{\Gamma\left(\frac{5}{3}\right)}.$$

Finally, denote by V the volume of the typical cell C of $\mathscr{V}_3$. Combining Equations (10) and (14) in [4] for dimension $d = 3$, and replacing the integration variables R and u used there by

$$R = \frac{\sin\beta}{\sin(\alpha+\beta)}, \quad u = \alpha,$$

the second volume moment can be written as

$$\mathrm{E}V^2 = \frac{24}{\lambda^2}\int_0^{\pi}\int_0^{\pi-\alpha}\frac{(\sin\alpha\sin\beta\sin(\alpha+\beta))^2\mathrm{d}\beta\mathrm{d}\alpha}{\left(\sin^3\alpha(2+\cos\beta(2+\sin^2\beta))+\sin^3\beta(2+\cos\alpha(2+\sin^2\alpha))\right)^2}. \tag{4}$$

In Section 3, (1),(2) as well as (4) serve as a starting point for further simplifications.

3 Results

In this section the new results of this paper are stated.

3.1 The second volume moment of the typical cell

The second volume moment of the typical cell C of the Poisson-Voronoi tessellation $\mathscr{V}_3$ in $\mathbb{R}^3$ is given by

$$\mathrm{E}V^2 = \frac{1}{\lambda^2}\left(-\frac{4}{3^5}\left(16\mathscr{A} - 3\pi^2\right) + \frac{8}{27}\sqrt{3}\,\pi\right) \tag{5}$$

with

$$\begin{aligned}\mathscr{A} &= 2\int_1^{\sqrt{3}} \frac{(1+z^2)\arctan z}{1-\frac{14}{9}z^2+z^4}\mathrm{d}z + \int_0^1 \frac{(3z-1)\operatorname{arctanh} z}{1-\frac{2}{3}z+z^2}\mathrm{d}z \\ &\approx 3.4954848920524811775,\end{aligned} \tag{6}$$

or, alternatively,

$$\begin{aligned}\mathscr{A} &= \frac{3}{2}\Bigg\{\left(\pi - \arctan\sqrt{2}\right)^2 - \frac{5}{72}\pi^2 - \frac{1}{4}(2\log 2 - \log 3)^2 - \frac{1}{8}\mathrm{Li}_2\left(\frac{1}{4}\right) \\ &\quad + \mathrm{Re}\Big(\mathrm{Li}_2(z_1) + \mathrm{Li}_2(z_2) - \mathrm{Li}_2(z_3) - \mathrm{Li}_2(z_4) - \mathrm{Li}_2(z_5) - \mathrm{Li}_2(\mathrm{i}z_5) \\ &\quad - \mathrm{Li}_2(z_6) - \mathrm{Li}_2(\mathrm{i}z_6)\Big)\Bigg\}\end{aligned} \tag{7}$$

with

$$z_{1,2} = \frac{1}{2} \pm \frac{1}{2\sqrt{6}} + \mathrm{i}\left(\frac{1}{2\sqrt{2}} \mp \frac{1}{2\sqrt{3}}\right), \quad z_{3,4} = \frac{1}{2} \pm \frac{1}{\sqrt{6}} + \mathrm{i}\left(\frac{1}{\sqrt{2}} \mp \frac{1}{2\sqrt{3}}\right),$$

$$z_5 = \frac{1+\sqrt{2}+\mathrm{i}\left(1-\sqrt{2}\right)}{2\sqrt{2}}, \quad z_6 = \frac{\sqrt{2}+1+\mathrm{i}\left(\sqrt{2}-1\right)}{3},$$

where

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\mathrm{d}t$$

for complex z. Note that in case $|z| \leq 1$, $\mathrm{Li}_2(z)$ can be also written as

$$\mathrm{Li}_2(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^2}.$$

3.2 Moments of the edge length distributions

The moments of the edge length distribution (the length L_s of the typical edge in $\mathscr{V}_s$) can be given in a similar form. In particular, the results are:

- in the linear section $\mathscr{V}_1$ (chord length):

$$\mathrm{E}L_1^4 = \left(\frac{3}{\pi\lambda}\right)^{\frac{4}{3}} \frac{4^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} \left(-\frac{1}{162}\left(16\mathscr{A}-3\pi^2\right)+\frac{1}{9}\sqrt{3}\pi\right), \tag{8}$$

$$\mathrm{E}L_1^7 = \left(\frac{3}{\pi\lambda}\right)^{\frac{7}{3}} \frac{4^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} \left(\frac{49}{8\cdot 3^5}\left(16\mathscr{A}-3\pi^2\right)-\frac{7}{60}\sqrt{3}\pi+\frac{1617}{25\cdot 2^6}\right), \tag{9}$$

$$\mathrm{E}L_1^{10} = \left(\frac{3}{\pi\lambda}\right)^{\frac{10}{3}} \frac{4^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} \left(-\frac{25025}{3^7\cdot 56}\left(16\mathscr{A}-3\pi^2\right)+\frac{135}{56}\sqrt{3}\pi-\frac{52641}{2^8\cdot 49}\right), \tag{10}$$

- in the planar section $\mathscr{V}_2$:

$$\mathrm{E}L_2^2 = \left(\frac{3}{\pi\lambda}\right)^{\frac{2}{3}} \frac{4^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \left(\frac{5}{162}\left(16\mathscr{A}-3\pi^2\right)+\frac{10}{81}\sqrt{3}\pi-\frac{5}{4}\right), \tag{11}$$

$$\mathrm{E}L_2^5 = \left(\frac{3}{\pi\lambda}\right)^{\frac{5}{3}} \frac{4^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \left(-\frac{50}{3^7}\left(16\mathscr{A}-3\pi^2\right)+\frac{125}{2\cdot 3^5}\sqrt{3}\pi-\frac{25}{48}\right), \tag{12}$$

$$\mathrm{E}L_2^8 = \left(\frac{3}{\pi\lambda}\right)^{\frac{8}{3}} \frac{4^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \left(\frac{1225}{3^8}\left(16\mathscr{A}-3\pi^2\right)-\frac{140}{81}\sqrt{3}\pi+\frac{41}{8}\right), \tag{13}$$

$$\mathrm{E}L_2^{11} = \left(\frac{3}{\pi\lambda}\right)^{\frac{11}{3}} \frac{4^{\frac{1}{3}}}{\Gamma\left(\frac{4}{3}\right)} \left(-\frac{5^3\cdot 3146}{3^{11}}\left(16\mathscr{A}-3\pi^2\right)+\frac{197175}{6^4 7}\sqrt{3}\pi-\frac{181005}{56^2}\right), \tag{14}$$

- and in the three-dimensional Poisson-Voronoi tessellation $\mathscr{V}_3$:

$$\mathrm{E}L_3^3 = \frac{3}{\pi\lambda}\left(-\frac{35}{3^5}\left(16\mathscr{A}-3\pi^2\right)-\frac{70}{81}\sqrt{3}\pi+\frac{35}{4}\right), \tag{15}$$

$$\mathrm{E}L_3^6 = \left(\frac{3}{\pi\lambda}\right)^2\left(-\frac{350}{3^7}\left(16\mathscr{A}-3\pi^2\right)+\frac{385}{3^5}\sqrt{3}\pi-\frac{49}{12}\right), \tag{16}$$

$$\mathrm{E}L_3^9 = \left(\frac{3}{\pi\lambda}\right)^3\left(\frac{7\cdot 5^4}{3^7}\left(16\mathscr{A}-3\pi^2\right)-\frac{520}{27}\sqrt{3}\pi+\frac{5925}{112}\right), \tag{17}$$

$$\mathrm{E}L_3^{12} = \left(\frac{3}{\pi\lambda}\right)^4\left(-\frac{385^2\cdot 689}{2\cdot 3^{13}}\left(16\mathscr{A}-3\pi^2\right)+\frac{11165}{36}\sqrt{3}\pi-\frac{53895}{2^6}\right). \tag{18}$$

Using the same reasoning as in Section 4, all higher moments $\mathrm{E}L_s^k$ for $k = 1,2,\ldots$ and $s = 1,2,3$ with k and s satisfying $k+2s \equiv 0 (\mathrm{mod}\ 3)$ can be obtained.

3.3 Moments of the linear contact distribution

The moments of the linear contact distribution function $H_l(r)$ are

$$\mathrm{E}R^3 = \frac{3}{\pi\lambda}\left(-\frac{1}{3^5}\left(16\mathscr{A}-3\pi^2\right)+\frac{2}{27}\sqrt{3}\,\pi\right), \tag{19}$$

$$\mathrm{E}R^6 = \left(\frac{3}{\pi\lambda}\right)^2\left(\frac{7}{3^6}\left(16\mathscr{A}-3\pi^2\right)-\frac{2}{45}\sqrt{3}\,\pi+\frac{77}{200}\right), \tag{20}$$

$$\mathrm{E}R^9 = \left(\frac{3}{\pi\lambda}\right)^3\left(-\frac{715}{2\cdot 3^8}\left(16\mathscr{A}-3\pi^2\right)+\frac{9}{14}\sqrt{3}\,\pi-\frac{17547}{15680}\right), \tag{21}$$

$$\mathrm{E}R^{12} = \left(\frac{3}{\pi\lambda}\right)^4\left(\frac{1616615}{2\cdot 3^{13}}\left(16\mathscr{A}-3\pi^2\right)-\frac{487}{108}\sqrt{3}\pi+\frac{71055}{4928}\right). \tag{22}$$

Numerical values for (5) and (8) - (22) are given in Table 1.

3.4 Conclusions

Comparison of (5) and (8) gives the identity

$$\mathrm{E}V^2 = \frac{\pi}{3}\left(\frac{4\pi}{3\lambda^2}\right)^{1/3}\Gamma\left(\frac{5}{3}\right)\mathrm{E}L_1^4,$$

see also [8] and [10, Prop. 3.4.5].

Furthermore, comparison of $(8),(9),(10)$ and $(19),(20),(21)$, respectively, gives

$$\mathrm{E}L_1^{k+1} = (k+1)\mathrm{E}L_1\,\mathrm{E}R^k. \tag{23}$$

This relationship is valid for all $k = 0,1,2,\ldots$ since it can simply be deduced from (3) by integration by parts. In this form, (23) does not seem to appear in the related literature so far.

4 Proofs

4.1 Proof of (5) (sketched)

The change of variables

$$\alpha = \arcsin\frac{2v}{1+v^2}+\arcsin\frac{2w}{1+w^2}, \qquad \beta = \arcsin\frac{2v}{1+v^2}-\arcsin\frac{2w}{1+w^2} \tag{24}$$

in (4) yields

$$\mathrm{E}V^2 = \frac{24}{\lambda^2}\int_0^1\int_0^v \frac{\left(1-v^2\right)^2\left(1+v^2\right)^3\left(1+w^2\right)^3\left(v^2\left(1-w^2\right)^2-w^2\left(1-v^2\right)^2\right)^2}{\left(1-v^2w^2\right)^6\left(v^2+3v^4+3w^2+v^2w^2\right)^2}\,\mathrm{d}w\,\mathrm{d}v.$$

Partial fraction decomposition, integration with respect to w and further simplification lead to

$$\mathrm{E}V^2 = \frac{24}{\lambda^2}\int_0^1 \left(\frac{P_1(v^2)}{v^5\left(1-\frac{2}{3}v^2+v^4\right)^4} + \frac{P_2(v^2)\,\operatorname{arctanh} v^2}{v^7\left(1-\frac{2}{3}v^2+v^4\right)^5} + \frac{vP_3(v^2)}{(1+3v^2)(3+v^2)\left(1-\frac{2}{3}v^2+v^4\right)^4} + \frac{v^5P_4(v^2)\arctan\sqrt{\frac{3+v^2}{1+3v^2}}}{\sqrt{(1+3v^2)^3(3+v^2)^3}\left(1-\frac{2}{3}v^2+v^4\right)^5}\right)\mathrm{d}v$$

with polynomial functions $P_j(t)$, $j=1,2,3,4$. The term containing $P_4(\cdot)$ equals

$$\int_1^{\sqrt{3}} \frac{P_5(z^2)\arctan z}{z^2\left(1-\frac{14}{9}z^2+z^4\right)^5}\,\mathrm{d}z,$$

where $P_5(t)$ is another polynomial, after the change of variable

$$v=\sqrt{\frac{z^2-3}{1-3z^2}},$$

and that containing $P_2(\cdot)$ can be simplified by substituting $v=\sqrt{z}$. Furthermore, integration by parts yields the recursion formula

$$\int_0^1 \frac{(p+qz)\operatorname{arctanh} z}{(1-\frac{2}{3}z+z^2)^j}\,\mathrm{d}z = \frac{3}{32(j-1)}\int_0^1 \frac{p(5-3z)+3q(1+z)}{(1+z)(1-\frac{2}{3}z+z^2)^{j-1}}\,\mathrm{d}z + \frac{3}{16(j-1)}\int_0^1 \frac{((7j-11)p+(j-1)q-3(j-2)(p-q)z)\operatorname{arctanh} z}{(1-\frac{2}{3}z+z^2)^{j-1}}\,\mathrm{d}z$$

for $j=2,3,\dots$ and rational p,q. In a similar fashion, one obtains

$$\int_1^{\sqrt{3}} \frac{(p+qz^2)\arctan z}{(1-\frac{14}{9}z^2+z^4)^j}\,\mathrm{d}z = \frac{1}{128(j-1)}\int_1^{\sqrt{3}} \frac{\left(pz(17-63z^2)+9qz(7-9z^2)\right)}{(1+z^2)(1-\frac{14}{9}z^2+z^4)^{j-1}}\,\mathrm{d}z + \frac{1}{128(j-1)}\int_1^{\sqrt{3}} \frac{\left((128j-111)p+63q-9(7-4j)(7p+9q)z^2\right)\arctan z}{(1-\frac{14}{9}z^2+z^4)^{j-1}}\,\mathrm{d}z + \frac{\pi\sqrt{3}}{96(j-1)}\left(\frac{3}{16}\right)^{j-1}(43p+45q) - \frac{\pi}{256(j-1)}\left(\frac{9}{4}\right)^{j-1}(23p+9q).$$

Partial fraction decomposition and gradual reduction of j lead finally to (5), where $\mathscr{A}$ is given in the form (6). □

Quite analogously, Equations (8) - (22) are obtained. If $k+2s\equiv 0(\mathrm{mod}\ 3)$ in (1) and $k\equiv 0(\mathrm{mod}\ 3)$ in (2), the transformation (24) leads to rational integrands, which can be simplified in the given way.

Table 1 Numerical values of the k-th moments of characteristics of the three-dimensional Poisson-Voronoi tessellation of unit intensity; V - cell volume, L_s - edge lengths in an s-dimensional section, R - linear contact distribution.

Characteristic	k	Value
V	2	1.179032437845
L_1	4	0.773693651532
L_1	7	1.736504680409
L_1	10	5.654144221662
L_2	2	0.403530946922
L_2	5	0.455961215101
L_2	8	0.996777692421
L_2	11	3.184976490816
L_3	3	0.245190266343
L_3	6	0.297110243132
L_3	9	0.656936829114
L_3	12	2.101954666119
R	3	0.281473260823
R	6	0.360999018049
R	9	0.822801328709
R	12	2.682083161431

4.2 Proof of (7)

Partial fraction decomposition of the denominator of the integrands of (6) into complex linear factors, use of

$$\operatorname{arctanh} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \quad |z|<1, \quad \arctan z = -\frac{\mathrm{i}}{2}\log\left(\frac{1+\mathrm{i}z}{1-\mathrm{i}z}\right),$$

$$\int \frac{\log(1+lz)}{z-z_0}\mathrm{d}z = \log(1+lz)\left(\log(l(z_0-z))-\log(1+lz_0)\right)+\mathrm{Li}_2\left(\frac{1+lz}{1+lz_0}\right)$$

for $l\in\{-1,+1,-\mathrm{i},+\mathrm{i}\}$ and complex z_0, and the dilogarithm identities

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(1-z) = \frac{\pi^2}{6}-\log(z)\log(1-z), \quad z\notin\{0,1\}$$

known as Eulers' reflexion formula,

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6}-\frac{1}{2}(\log(-z))^2, \quad z\notin[0,1),$$

$$\mathrm{Li}_2(-z)+\mathrm{Li}_2(z) = \frac{1}{2}\mathrm{Li}_2\left(z^2\right)$$

and

$$\mathrm{Li}_2(z)+\mathrm{Li}_2(\bar{z}) = 2\mathrm{Re}\left(\mathrm{Li}_2(z)\right)$$

lead to (7). □

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