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Vorwort

Das Tätigkeitsfeld des Fraunhofer-Instituts für Techno- und Wirtschaftsmathematik ITWM umfasst anwendungsnahe Grundlagenforschung, angewandte Forschung sowie Beratung und kundenspezifische Lösungen auf allen Gebieten, die für Techno- und Wirtschaftsmathematik bedeutsam sind.

In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe sollen sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen werden.

Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

Fraunhofer Institut Techno- und Wirtschaftsmathematik

HOMOGENIZATION IN ELASTO-PLASTICITY

Technical Report

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Contents

Abstract

The theory of the two-scale convergence was applied to homogenization of elasto-plastic composites with a periodic structure and exponential hardening law. The theory is based on the fact that the elastic as well as the plastic part of the stress field two-scale converges to a limit, which is factorized by parts, depending only on macroscopic characteristics, represented in terms of corresponding part of the homogenised stress tensor and only on stress concentration tensor, related to the micro-geometry and elastic or plastic micro-properties of composite components. The theory was applied to metallic matrix material with Ludwik and Hocket-Sherby hardening law and pure elastic inclusions in two numerical examples. Results were compared with results of mechanical averaging based on the self-consistent methods.

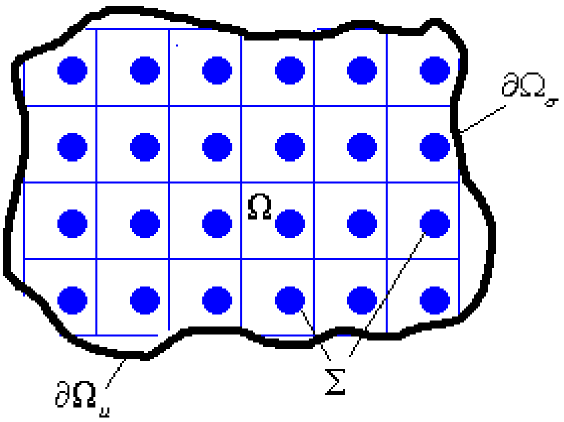
Keywords: Multiscale structures, Asymptotic homogenization, Nonlinear energy

1 Introduction

Since the 70-s, the homogenization technique was properly used for obtaining effective elastic properties of composites, and the homogenized macro-stress-fields. Although, the homogenized stress-field looses all local effects, e.g. local stress concentrations, which can reason the fracture or plastic deformations occurrence. In the 80-s, the first approximation to the micro-stress-field was derived on the pure formal way, from the properties of the components, micro-geometry of the composite and the applied macro-loads. Convergence of the micro-stresses, as the structure period tends to zero, was proved by the two-scale homogenization by Allaire in 1992. The present work is based on the fact that for elastic and some nonlinear solids , for which stress tensor is a homogeneous function of the deformation gradient, this limit is exactly the first term of the micro-stress-field approximation, which is represented by a product of the homogenized stress-tensor, depending only on macro-geometry and boundary conditions, and the so-called stress-concentration tensor, related only to the micro-geometry and mechanical parameters of composite-components.
The paper provides the complete theoretical back-ground and a detailed formal algorithm for a class of elasto-plastic composites with a periodic micro-structure. Sec. 2 provides an auxiliary tool for the main result, Sections 7, 8, of this manuscript. Here the well-known convergence results for a class of non-linear elliptic PDE's are recalled. In the next two Sections, 3 and 4, we repeat elements of strength analysis and homogenization of strength from [Orlik, 2005]. We found in this work, that these theory can be one-to-one transfered and applied to the yield analysis and homogenization of the yield limit, which is the subject of this work. This finding is new and can, together with examples of special elasto-plastic models from Sections 7, 8, be a subject of a journal paper in an appropriate journal later on. Algorithm for homogenization of elastic properties prescribed in Sec. 6 is rather standard. Nevertheless we recalled it in the preprint to once have a complete detailed homogenization algorithm for elasto-plastic composites.
The theory of the two-scale convergence was applied in Sections 7, 8 to homogenization of elasto-plastic metallic composites with Ludwik or Hocket-Sherby hardening matrix material and pure elastic inclusions, creating the periodic structure. Such materials were the subject of an industrial project from the automobile industry. A numerical example with randomly situated ball-shaped inclusions with normally distributed radiuce and volume fraction of inclusions $c = 10\%$ was considered in Sec. 9. The asymptotic homogenization was performed with the help of the commercial finite element software ABAQUS and python-scripting. Results of the asymptotic homogenization were compared with results of mechanical averaging based on the self-consistent methods, recalled in Sec. 10, and are given in tables below.

2 Elements of homogenization for nonlinear elliptic PDE's. Existence, and convergence

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a periodic structure with a period ϵY. Consider the following problem



$$-\mathbf{div}\sigma^{\epsilon}(x) = f(x), \;\; \sigma^{\epsilon}(x) = g(\frac{x}{\epsilon}, e^{\epsilon}(x)), \;\; x \in \Omega \tag{1}$$

with some appropriate boundary conditions. Furthermore, let $e(x): \;\Omega \rightarrow \mathbb{R}^{3\times3}$, $g(e): \;\mathbb{R}^{3\times3} \rightarrow \mathbb{R}^{3\times3}$ be an arbitrary non-linear function.

Example 1 *In non-linear elasticity $\sigma = g(e) = \frac{dW(e)}{de}$, where W is the energy function.*

Assumption 2 *(i) $e \rightarrow g(\xi, e)$ is continuous $\forall\xi$;*

(ii) $\xi \rightarrow g(\xi, e)$ is measurable and Y-periodic $\forall e$;

(iii) $0 \leq c|e|^k \leq g(\xi, e)\cdot e$ with $0 < c$ and $k > 1$, $\forall\xi$;

(iv) $|g(\xi, e)| \leq C[1 + |e|^{k-1}]$, for $0 < C$, $\forall\xi$;

(v) $g(\xi, e)$ is strictly monotone in e, i.e., $\forall\xi$,
$[g(\xi, e_1) - g(\xi, e_2)]\cdot(e_1 - e_2) > 0 \; \forall e_1 \neq e_2$.

According to [1], if Assumptions 2 on the function g are satisfied, then, the Dirichlet problem

$$-\mathbf{div}\,(g(x, \nabla u(x))) = f(x),$$

has a unique solution $u \in H^1(\Omega, \partial\Omega_u)$, $g(x, \nabla u(x)) \in L^2(\Omega)$.

Furthermore, the application of the two-scale homogenization technique allows under certain conditions for the complete separation of problems on the periodicity cell and the homogenized problem in the macro-domain. Therefore, the following assumption should be additionally required:
Assumption 3 *(vi) $g(\xi, e)$ is homogeneous of the order $k-1$ w.r.t. e, i.e.,*

$$g(\xi, \lambda e) = \lambda^{k-1} g(\xi, e),\ \lambda \in \mathbb{R}^{n\times n}.$$

The following Theorem can be found as Th.3.5 in [1].
Theorem 4 *For any $f \in L^2(\Omega)$, the system*

$$-\mathbf{div}\ \ g\left(\tfrac{x}{\epsilon}, \nabla u^\epsilon\right) = f(x),\ x \in \Omega$$

with g satisfying Assumptions 2, 3, converges to

$$-\mathbf{div} \int_Y g(\xi, \nabla_\xi(\xi_p I + N_p(\xi))\nabla_x u^{(0)}(x))d\xi = f(x),$$

where $N_p(\xi)$ has to be determined from the auxiliary periodic problem:

$$-\mathbf{div}_\xi\ \ g\left(\xi, \nabla_\xi(\xi_p I + N_p(\xi))\right) = 0,\ \xi \in Y.$$

3 Elements of yield analysis

For a bounded stress field $\sigma_{ij}(y)$, any local yield condition for micro-stresses at a point $y \in \Omega$ can be written in the form $\Lambda\left(\sigma(y), y\right) < 1$, where $\Lambda \in C(\mathbb{R}^{n\times n}, \mathcal{M}(\Omega))$ is a normalised equivalent stress function, a material characteristic, which is non-negative and positively homogeneous of the order +1 w.r.t. σ. Here $\mathcal{M}(\Omega)$ is the space of all bounded functions on Ω.
Example 5 *For most materials Λ can be associated with the von Mises equivalent stress*

$$\Lambda_M(\sigma(y), y) = \{[(\sigma_1(y) - \sigma_2(y))^2 + (\sigma_2(y) - \sigma_3(y))^2 + (\sigma_3(y) - \sigma_1(y))^2]\}^{1/2}/(\sqrt{2}\sigma_r(y)), \qquad (2)$$

where σ_r is a known uniaxial yield limit of the material.
Usually plastification of body, i.e. occurring of plastic deformations starts in one of its points, so that the (initial) local yield condition for the whole body is then

$$\sup_{y\in\Omega} \Lambda\left(\sigma(y), y\right) < 1. \qquad (3)$$

Such a local yield condition, however, is generally not applicable to unbounded stress fields since the conditions will predict plastic zones under virtually any singular stress field.

For more general, especially singular stress fields, e.g. belonging to $L_2(\Omega)$, a (point) non-local yield condition $\Lambda^\odot(\sigma; y) < 1$ can be applied. Here $\Lambda^\odot(\sigma; y)$ is a normalised equivalent stress functional, which is defined on the tensor-functions $\sigma_{ij} \in L_2(\Omega)$ and is non-negative positively homogeneous w.r.t. σ.

Particularly $\Lambda^\odot$ can be related to a weighted averaging of $\sigma_{ij}(x)$, $x \in \Omega$ in some neighborhood of the point y,

$$\Lambda^\odot(\sigma; y) = \Lambda\left(\sigma^\odot(y), y\right), \quad \sigma^\odot_{ij}(y) = \int_\Omega w_{ijkl}(y, x)\sigma_{kl}(x)dx, \qquad (4)$$

where $\sigma^\odot_{ij} \in C(\bar\Omega)$ is an auxiliary *non-local* stress tensor, and the weight $w \in C(\bar\Omega, L^2(\Omega))$ is a material characteristics, such as $\int_\Omega w_{ijkl}(y, x)dx = \delta_{ij}\delta_{kl}$. Then the non-local yield condition for the whole body is

$$\sup_{y\in\Omega} \Lambda\left(\sigma^\odot, y\right) < 1 \qquad (5)$$

Example 6 *(i) If* $w_{ijkl}(y,x) = \delta_{ij}\delta_{kl} \begin{cases} \frac{3}{4\pi d^3}, & |x-y| < d \\ 0, & |x-y| \geq d \end{cases}$ *, where* d *is a material constant, then* $\sigma^{\odot}_{ij}(y) = \frac{3}{4\pi d^3}\int_{|x-y|<d}\sigma_{ij}(x)dx.$
(ii) If $w_{ijkl}(y,x) = \delta(x-y)\delta_{ij}\delta_{kl}$*, where* $\delta(x-y)$ *is the Dirac function, then* $\sigma^{\odot}_{ij}(y) = \sigma_{ij}(y)$*, and the non-local yield-condition coincides with the local one.*

4 Homogenization of yield conditions

This subsection can be found in more details in [Orlik, 2005]. The theory was developped for the strength condition, but can be generalized without any restrictions to the yield condition.
In a periodic medium, all physical fields becomes dependent on the scaling parameter ϵ. Especially, yield condition (3) becomes

$$\sup_{y\in\Omega}\Lambda^{\epsilon}(\sigma^{\epsilon}, y) < 1. \tag{6}$$

Suppose,

$$\Lambda^{\epsilon}(\sigma^{\epsilon}, y) = \Lambda(\sigma^{\epsilon}, \frac{y}{\epsilon}). \tag{7}$$

Our aim is to derive a macro-yield condition like (5), where the homogenized yield function $\hat{\Lambda}$ is a function of the homogenized stress $\hat{\sigma}_{ih}(x)$ and the composite micro-characteristics only. Let $|\sigma|$ denotes a mytrix norm for a tensor σ_{ij}.

Proposition 7 (homogenization of local normalised equivalent yield function) *Let a tensor function sequence* $\sigma^{\epsilon}(y) \in C(\bar{\Omega})$ *converge to a tensor function* $\sigma^0(y,\zeta) \in C(\bar{\Omega}, C_{per}(Y))$ *uniformly w.r.t.* y *as* $\epsilon \to 0$*, i.e.,*

$$\lim_{\epsilon\to 0}\sup_{y\in\Omega}|\sigma^{\epsilon}(y) - \sigma^0(y, \frac{y}{\epsilon})| = 0 \tag{8}$$

and $\Lambda \in C(\mathbb{R}^{n\times n}, \mathcal{M}_{per}(Y))$*. Then,*

$$\lim_{\epsilon\to 0}\sup_{y\in\Omega}\left|\Lambda\left(\sigma^{\epsilon}(y), \frac{y}{\epsilon}\right) - \Lambda\left(\sigma^0(y, \frac{y}{\epsilon}), \frac{y}{\epsilon}\right)\right| = 0 \tag{9}$$

If

$$\sigma^0(x\zeta) = B_{ijkl}(\zeta)\hat{\sigma}_{kl}(y), \zeta), \tag{10}$$

then

$$\lim_{\epsilon\to 0}\sup_{y\in\Omega}\Lambda\left(\sigma^{\epsilon}(y), \frac{y}{\epsilon}\right) \leq \sup_{y\in\Omega}\hat{\Lambda}(\hat{\sigma}(y)),\ \hat{\Lambda}(\hat{\sigma}(y)) := \sup_{\zeta\in Y}\Lambda(B_{ijkl}(\zeta)\hat{\sigma}_{kl}(y), \zeta).$$

and the limit sufficient local macro-yield condition is

$$\sup_{y\in\Omega}\hat{\Lambda}(\hat{\sigma}(y)) < 1.$$

Let us consider for a periodic medium the limits of *non-local* micro–yield conditions $\sup_{y\in\Omega}\Lambda^{\odot\epsilon}(\sigma^{\epsilon}; y) < 1$ as $\epsilon \to 0$. Representation (4) becomes

$$\begin{aligned} \Lambda^{\odot\epsilon}(\sigma^{\epsilon}; y) &= \Lambda^{\epsilon}\left(\sigma^{\odot\epsilon}(y), y\right) \\ \sigma^{\odot\epsilon}_{ij}(y) &= \int_{\Omega} w^{\epsilon}_{ijkl}(y,x)\sigma^{\epsilon}_{kl}(x)dx, \quad \forall y \in \Omega. \end{aligned}$$

Suppose the function Λ^ϵ has form (7). Let further
$w^\epsilon_{ijkl}(y,x) = w_{ijkl}(y, \frac{y}{\epsilon}, x, \frac{x}{\epsilon})$. Then

$$\begin{aligned} \Lambda^\epsilon(\sigma^{\odot\epsilon}, y) &= \Lambda(\sigma^{\odot\epsilon}(y), \frac{y}{\epsilon}), \\ \sigma^{\odot\epsilon}_{ij}(y) &= \int_\Omega w_{ijkl}(y, \frac{y}{\epsilon}, x, \frac{x}{\epsilon})\sigma^\epsilon_{kl}(x)dx. \end{aligned} \tag{11}$$

Lemma 8 *Let $\sigma^\epsilon \in L^2(\Omega)$ be a sequence of tensor functions $\sigma^\epsilon(x)$ two–scale converging to a tensor $\sigma^0 \in L^2(\Omega \times Y)$ as $\epsilon \to 0$. Let $w \in C(\bar{\Omega}, C_{per}(Y, L^2(\Omega, C_{per}(Y))))$. Then the sequence*

$$\tilde{\sigma}^{\odot\epsilon}(y,\zeta) := \int_\Omega w(y,\zeta,x,\frac{x}{\epsilon})\sigma^\epsilon(x)dx, \tag{12}$$

is bounded in $C(\bar{\Omega}, C_{per}(Y))$ and strongly converges in this space to

$$\sigma^{\odot 0}(y,\zeta) = \frac{1}{|Y|}\int_\Omega\int_Y w(y,\zeta,x,\xi)\sigma^0(x,\xi)d\xi dx. \tag{13}$$

Proposition 9 *Let $\sigma^\epsilon \in L^2(\Omega)$ be a sequence of tensor functions $\sigma^\epsilon(x)$ two–scale converging to a tensor $\sigma^0 \in L^2(\Omega \times Y)$. Suppose, $w \in C(\bar{\Omega}, C_{per}(Y, L^2(\Omega, C_{per}(Y))))$. Then, the sequence $\sigma^{\odot\epsilon}(y)$ given by (11) is bounded in $C(\bar{\Omega})$ and converges to the tensor function $\sigma^{\odot 0}(y, \frac{y}{\epsilon})$, given by (13), uniformly w.r.t. y as $\epsilon \to 0$, i.e.,*

$$\lim_{\epsilon\to 0}\sup_{y\in\bar{\Omega}}\left|\sigma^{\odot\epsilon}(y) - \sigma^{\odot 0}(y, \frac{y}{\epsilon})\right| = 0.$$

Applying Proposition 7 to the non-local stresses (13) and taking into account their convergency proved in Proposition 9, we arrive at the following proposition on homogenization of non-local strength conditions.
Proposition 10 *Let $\sigma^\epsilon \in L^2(\Omega)$ be a sequence of tensor functions two–scale converging to a tensor $\sigma^0 \in L^2(\Omega \times Y)$. Suppose the non-local stress $\sigma^{\odot\epsilon}(y)$ is given by (11) with the weight $w \in C(\bar{\Omega}, C_{per}(Y, L^2(\Omega, C_{per}(Y))))$. Suppose $\Lambda \in C(\mathbb{R}^{n\times n}, \mathcal{M}_{per}(Y))$. Then*

$$\lim_{\epsilon\to 0}\sup_{y\in\Omega}\left|\Lambda\left(\sigma^{\odot\epsilon}(y), \frac{y}{\epsilon}\right) - \Lambda\left(\sigma^{\odot 0}(y, \frac{y}{\epsilon}), \frac{y}{\epsilon}\right)\right| = 0,$$

where $\sigma^{\odot 0}(y,\zeta)$ is given by (13). If σ^0 is expressed by (10), then

$$\begin{aligned} \lim_{\epsilon\to 0}\sup_{y\in\Omega}\Lambda\left(\sigma^{\odot\epsilon}(y), \frac{y}{\epsilon}\right) &\le \sup_{y\in\Omega}\hat{\Lambda}^\odot(\hat{\sigma}; y), \\ \text{where} \quad \hat{\Lambda}^\odot(\hat{\sigma}; y) &:= \sup_{\zeta\in Y}\Lambda\left(\int_\Omega \hat{w}_{ijkl}(y,\zeta,x)\hat{\sigma}_{kl}(x)dx, \zeta\right), \\ \hat{w}_{ijkl}(y,\zeta,x) &= \frac{1}{|Y|}\int_Y w_{ijpq}(y,\zeta,x,\xi)A_{pqkl}(\xi)d\xi. \end{aligned} \tag{14}$$

Finally the limit sufficient non-local macro-yield condition is $\sup_{y\in\Omega}\hat{\Lambda}^\odot(\hat{\sigma}; y) < 1$.

5 Statement of the problem

We consider elasto-plastic metallic composites with pure elastic inclusions, creating the periodic structure. It is assumed in elasto-plasticity that the rate of the deformation tensor can be additively decomposed into an elastic and plastic part:

$$\dot{e} = \dot{e}_{el} + \dot{e}_{pl}$$

$$\boldsymbol{\sigma} = \mathbb{L}_0^{(2)} : e_{el},$$

where $\mathbb{L}_0^{(2)}$ is the fourth order elasticity tensor.
For isotropic materials a yield surface can be expressed as

$$F(\boldsymbol{\sigma}, \overline{e}_{pl}) = \frac{3}{2}\boldsymbol{s} : \boldsymbol{s} - \sigma_y^2(\overline{e}_{pl}) = 0,$$

where the equivalent von Mises stress is defined as

$$\sigma_{eq} = \sqrt{\frac{3}{2}\boldsymbol{s} : \boldsymbol{s}},$$

the equivalent plastic strain is defined as

$$\overline{e}_{pl} = \sqrt{\frac{2}{3} e_{pl} : e_{pl}}$$

and the uni-axial yield stress σ_y is supposed in this report to be taken in the form of Ludwik or Hocket-Sherby hardening:

$$\sigma_y^\varepsilon = \begin{cases} \sigma_{pl}(x,\xi) = S_0(\xi) + K(\xi)\overline{e}_{pl}^n(x,\xi) \text{ Ludwik } 1D \\ \sigma_{pl}(x,\xi) = b(\xi) - (b(\xi) - a(\xi))\exp(-m(\xi)\cdot(\overline{e}_{pl})^n(x,\xi)) \text{ Sherby } 1D \end{cases},$$

where $\xi = \frac{x}{\varepsilon}$.

According to the postulate of Drucker, which is a valid approximation for metals, the yield surface is convex and the plastic rate of the deformation tensor is orthogonal to the yield surface (flow rule)

$$\dot{\overline{e}}_{pl}^\varepsilon = \dot{\lambda}\frac{\partial F^\varepsilon}{\partial \boldsymbol{\sigma}^\varepsilon} = 2\dot{\lambda}\sigma_y^\varepsilon, \quad \dot{\lambda} \geq 0,$$

where the plastic consistency parameter $\dot{\lambda}$ is positive for plastic deformations and set to zero for elastic deformation.

Keeping in mind that

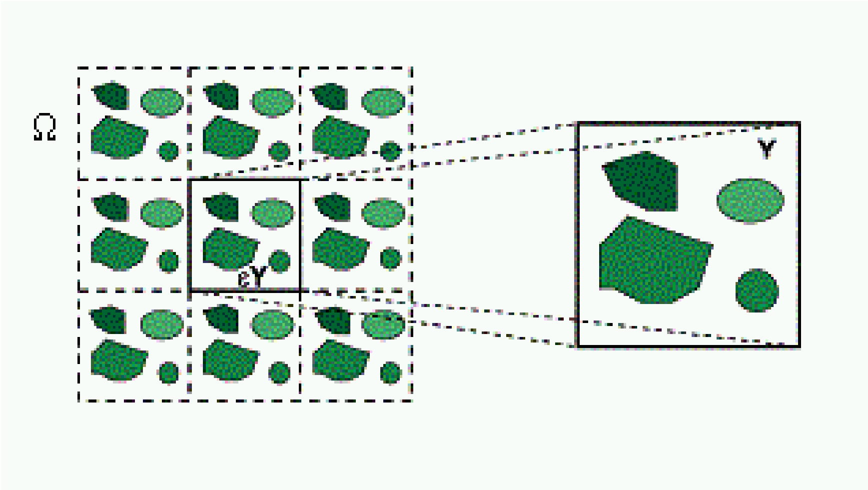
$$\dot{\overline{e}}_{pl}^\varepsilon = \frac{\Delta\overline{e}_{pl}^\varepsilon}{\Delta t},$$

we can obtain the scalar equation for the von Mises stress-strain in the form:

$$\Delta\overline{e}_{pl}^\varepsilon = \Delta t \cdot 2\dot{\lambda}\sigma_y^\varepsilon(\overline{e}_{pl}^\varepsilon).$$

6 Homogenization of elastic properties $E(\xi)$ and $\nu(\xi)$

In this Section we refere to the well known homogenization technique for pure elasticity given, e.g., in PhD Orlik, 2000, but recall this in all algorithmic detailes.



6.1 Statement of the elastic problem with a periodic structure

Assume that a solid $\Omega \subset \mathbb{R}^n$ has ϵY-periodic structure, where ϵ is a small scaling parameter. We recall the governing equations for the elasto-plastic model. The equilibrium equation in the variational formulation has the form:

$$\boxed{\int_V \sigma^\epsilon_{ji} \delta \frac{\partial v^\epsilon_i}{\partial x_j} dV = \int_{S_t} \phi_i \delta v_i dS,} \tag{15}$$

where σ^ϵ_{lj} are components of the Cauchy stress tensor, ϕ_i is the traction on the surface. The constitutive law for this model can be summarized as the following:

$$\sigma^\epsilon_{ij}(x) = L^\epsilon_{ijkl}(x) e^\epsilon_{kl}(x), \tag{16}$$

$$e^\epsilon_{kl}(x) = \frac{1}{2}\left(\frac{\partial u^\epsilon_k}{\partial x_l} + \frac{\partial u^\epsilon_l}{\partial x_k}\right), \tag{17}$$

$$L^\epsilon_{ijkl}(x) := L_{ijkl}(x, \frac{x}{\epsilon}). \tag{18}$$

Here L_{ijkl} are components of the elastic stiffness tensor, e^ϵ_{kl} are components of the elastic strain tensor, u^ϵ_k are components of the displacement vector. The small parameter ϵ denotes the period of the structure and as a subscript denotes the relation of all physical values on the micro-structure.

6.2 Formal asymptotical expansion

We study the asymptotic behaviour of the displacement field $u(x)$ with respect to a small fixed ratio ε. According to [2] we are looking for asymptotics of the solution to $(15) - (18)$ of the form

$$u^\epsilon_j(x) = \sum_{m=0}^{\infty} \varepsilon^m v^{(m)}_j(x, \frac{x}{\varepsilon}), \qquad j = 1...3, \tag{19}$$

where $v^{(m)}_j(x, \xi)$ are functions periodic with respect to ξ. After the substitution of (19) into the equilibrium equations and after using these equations for the terms of orders $\varepsilon^{-2}, \varepsilon^{-1}, \varepsilon^0$ separately, we get the following conditions

$$\mathcal{F}_{i\xi\xi} v^{(0)} = 0, \tag{20}$$

$$\mathcal{F}_{i\xi\xi} v^{(1)} = -\mathcal{F}_{i\xi x} v^{(0)} - \mathcal{F}_{ix\xi} v^{(0)}, \tag{21}$$

$$\mathcal{F}_{i\xi\xi}v^{(2)} = -\mathcal{F}_{i\xi x}v^{(1)} - \mathcal{F}_{ix\xi}v^{(1)} - \mathcal{F}_{ixx}v_i^{(0)}, \tag{22}$$

where we use the notations

$$\mathcal{F}_{i\alpha\beta}w(x,\xi) := \frac{\partial}{\partial\alpha_j}\left(L_{ijkl}(x,\xi)\frac{1}{2}\left(\frac{\partial w_k}{\partial\beta_l} + \frac{\partial w_l}{\partial\beta_k}\right)\right), \tag{23}$$

Here x and ξ are regarded as independent variables; $\alpha,\ \beta$ are x or ξ.
Consequently, we have the following boundary conditions

$$v^{(0)}|_{\partial\Omega_u} = 0,\ \ v^{(1)}|_{\partial\Omega_u} = 0, \tag{24}$$

$$\left[L_{ijkl}(x,\xi)\frac{1}{2}\left(\frac{\partial v_k^{(1)}(x,\xi)}{\partial\xi_l} + \frac{\partial v_l^{(1)}(x,\xi)}{\partial\xi_k} + \frac{\partial v_k^{(0)}(x)}{\partial x_l} + \frac{\partial v_l^{(0)}(x)}{\partial x_k}\right)\right] n_j(x)|_{\partial\Omega_\sigma} = \phi_i(x). \tag{25}$$

6.3 Equations on a cell and averaged equation

Equation (20) implies

$$v_j^{(0)}(x,\xi) := v_j^{(0)}(x). \tag{26}$$

where $v_j^{(0)}(x)$ are arbitrary functions independent of ξ.

Now, we can split solution of equation (21) as following (see e.g.,[2])

$$v_j^{(1)}(x,\xi) := N_{jp}^q(\xi)\frac{\partial v_p^{(0)}(x)}{\partial x_q}, \tag{27}$$

where the unknown periodic functions $N_{jp}^q(\xi)$ must be such that the first term on right hand side of (27) satisfies equation (21) with the part of its original right-hand side, depending on $v^{(0)}$. Such substitution reduces the initial problem to the cell problems for the periodic functions $N_{jp}^q(\xi)$:

$$\mathcal{F}_{i\xi\xi}\left(N_{.p}^q(\xi) + \xi_q\delta_{.p}\right) = 0, \tag{28}$$

i.e.,

$$\boxed{\int_Y \left[L_{ihjk}(x,\xi)\left(\frac{\partial\left(N_{jp}^q(\xi) + \xi_q\delta_{jp}\right)}{\partial\xi_k} + \frac{\partial\left(N_{kp}^q(\xi) + \xi_q\delta_{kp}\right)}{\partial\xi_j}\right)\right]\frac{\partial w_i(\xi)}{\partial\xi_h}d\xi = 0.}$$

The solvability condition for equation (22) for $v^{(2)}$, gives the averaged equation:

$$\frac{1}{|Y|}\int_Y \left(\mathcal{F}_{ix\xi}v^{(1)} + \mathcal{F}_{ixx}v^{(0)}\right)d\xi = 0. \tag{29}$$

If we multiply this equation by some test function $w \in \mathbf{H^1}(\Omega, \partial\Omega_u)$, and perform the partial integration by taking in consideration boundary condition (25), we obtain

$$\frac{1}{|Y|}\int_\Omega \frac{\partial w_i(x)}{\partial x_h}dx\int_Y \left[L_{ijkl}(x,\xi)\frac{1}{2}\left(\frac{\partial v_k^{(1)}(x,\xi)}{\partial\xi_l} + \frac{\partial v_l^{(1)}(x,\xi)}{\partial\xi_k} + \frac{\partial v_k^{(0)}(x)}{\partial x_l} + \frac{\partial v_l^{(0)}(x)}{\partial x_k}\right)\right]d\xi$$
$$= \int_{\partial\Omega_\sigma}\phi_i(x)w_i(x)ds. \tag{30}$$

By taking into consideration the substitution (27), we can formally interpret (30) as the following averaged equation for $v_j^{(0)}(x)$:

$$\boxed{\int_\Omega \left[L^H_{ijkl}(x,\xi)e^{(0)}_{kl}\right)\right] \frac{\partial w_i(x)}{\partial x_j}dx = \int_{\partial\Omega_\sigma} \phi_i(x)w_i(x)ds,} \tag{31}$$

where the averaged coefficients and plastic stresses are given by the following expressions:

$$L^H_{ihpq}(x) := \frac{1}{2|Y|}\int_Y L_{ijhk}(x,\xi)\left(\frac{\partial\Big(N^q_{jp}(\xi)+\xi_q\delta_{jp}\Big)}{\partial\xi_k} + \frac{\partial\Big(N^q_{kp}(\xi)+\xi_q\delta_{kp}\Big)}{\partial\xi_j}\right)d\xi, \tag{32}$$

6.4 Algorithm

The averaging problem can be numerically solved in the following way.
1. Solve equation (87) on a cell in the class of periodic matrix-valued functions with periodicity conditions on the cell boundary by the finite element method (FEM).
2. Calculate the effective coefficients and averaged plastic stress by equations (89) - (91).
3. Solve by the FEM averaged equation (31) in Ω.
4. Calculate the first approximation to the micro-stresses as the following:

$$\sigma^{\epsilon}_{ih}(x) \approx \sigma^{(0)}_{ih}(x, \frac{x}{\epsilon}) + O(\epsilon), \tag{33}$$

$$\sigma^{(0)}_{ih}(x) = L_{ihjk}(x, \frac{x}{\epsilon})\frac{1}{2}\left[\left(\frac{\partial\Big(N^q_{jp}(\xi) + \xi_q\delta_{jp}\Big)}{\partial\xi_k} + \frac{\partial\Big(N^q_{kp}(\xi) + \xi_q\delta_{kp}\Big)}{\partial\xi_j}\right)\Big|_{\xi=\frac{x}{\epsilon}}\ \frac{\partial v^{(0)}_p(x)}{\partial x_q}\right].$$

6.5 Periodicity conditions for the cell problem

For solving equation (87) we fix q. Then, we have

$$\mathbf{N}^q(\xi) = \begin{pmatrix} N_{11q} & N_{12q} & N_{13q} \\ N_{21q} & N_{22q} & N_{23q} \\ N_{31q} & N_{32q} & N_{33q} \end{pmatrix},$$

$$\begin{aligned}(\mathbf{N}^q(\xi)\ +\ \xi_q\mathbf{I}) \quad &:= (N_{jpq}(\xi)) + \delta_{jp}\xi_q)_{j,p=(1,\ldots,3)} \\ &= \begin{pmatrix} N_{11q}+\xi_q & N_{12q} & N_{13q} \\ N_{21q} & N_{22q}+\xi_q & N_{23q} \\ N_{31q} & N_{32q} & N_{33q}+\xi_q \end{pmatrix} =: \begin{pmatrix} \mathbf{Y}^1_q \\ \mathbf{Y}^2_q \\ \mathbf{Y}^3_q \end{pmatrix}.\end{aligned} \tag{34}$$

Then, for each $q = 1, 2, 3$, three boundary value problems of elasticity have to be solved:

$$\begin{aligned}\mathcal{F}_{\xi\xi}(\mathbf{Y}^1) &= 0 \\ \mathcal{F}_{\xi\xi}(\mathbf{Y}^2) &= 0 \\ \mathcal{F}_{\xi\xi}(\mathbf{Y}^3) &= 0,\end{aligned} \tag{35}$$

where $\mathcal{F}_{\xi\xi}$ is the operator defined by the left hand-side of (23) and each of the $\mathbf{Y}^p$, $p = 1, ..., 3$ can be interpreted as a vector of velocities. The boundary conditions for each of the above equations, i.e., $\mathbf{Y}^p$, $p = 1, ..., 3$ can be obtained by using the periodicity conditions for vector-function $\mathbf{N}^q(\xi)$.

Periodicity conditions:
Let us introduce a coordinate system w.r.t. ξ by taking its origin in the middle of the unit cell.
Let $d = \pm\frac{1}{2}$. We introduce a mapping $S_i\xi$ w.r.t. the surface $\xi_i = 0$ as the following:

$$S_i\xi = ((-1)^{\delta_{i1}}\xi_1, (-1)^{\delta_{i2}}\xi_2, (-1)^{\delta_{i3}}\xi_3),$$

where δ_{ij} is the Kronecker symbol, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Since $N_j \in H^1_{per[0]}(Y)$,

$$N_j(S_i\xi)\ |_{\xi_i=d} = N_j(\xi)\ |_{\xi_i=d}\ . \tag{36}$$

However, we can not claim that σ_{ih} would be equal on the opposite sides of the cell.

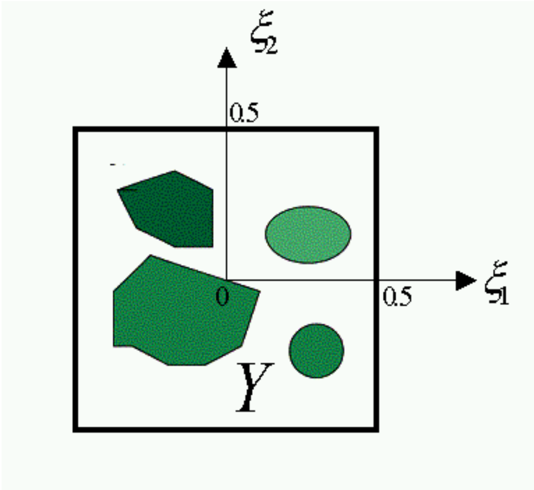


Figure 1 Coordinate system on the unit cell

Let us now express the boundary conditions for equations (35) in terms of displacements. For this end, recall that for each fixed q and $p = 1, 2, 3$, $v_j^{pq}(\xi) = Y_{jq}^p$, where v_1^p, v_2^p, v_3^p are components of the displacement vector on the unit cell. Comparing this expression with (34), we obtain
for $q = 1$

$$\begin{aligned} v_1^{11}(\xi) = N_{111}(\xi) + \xi_1, \quad & v_2^{11}(\xi) = N_{121}(\xi), \quad & v_3^{11}(\xi) = N_{131}(\xi) \\ v_1^{21}(\xi) = N_{211}(\xi), \quad & v_2^{21}(\xi) = N_{221}(\xi) + \xi_1, \quad & v_3^{21}(\xi) = N_{231}(\xi) \\ v_1^{31}(\xi) = N_{311}(\xi), \quad & v_2^{31}(\xi) = N_{321}(\xi), \quad & v_3^{31}(\xi) = N_{331}(\xi) + \xi_1 \end{aligned}$$

$q = 1$	$p = 1$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5} + 1$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q = 1$	$p = 2$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5} + 1$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q = 1$	$p = 3$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5} + 1$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

for $q = 2$

$$\begin{aligned} v_1^{12}(\xi) = N_{112}(\xi) + \xi_2, \quad & v_2^{12}(\xi) = N_{122}(\xi), \quad & v_3^{12}(\xi) = N_{132}(\xi) \\ v_1^{22}(\xi) = N_{212}(\xi), \quad & v_2^{22}(\xi) = N_{222}(\xi) + \xi_2, \quad & v_3^{22}(\xi) = N_{232}(\xi) \\ v_1^{32}(\xi) = N_{312}(\xi), \quad & v_2^{32}(\xi) = N_{322}(\xi), \quad & v_3^{32}(\xi) = N_{332}(\xi) + \xi_2 \end{aligned}$$

$q=2$	$p=1$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5} + 1$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q=2$	$p=2$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5} + 1$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q=2$	$p=3$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5} + 1$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

for $q=3$

$$v_1^{13}(\xi) = N_{113}(\xi) + \xi_3, \quad v_2^{13}(\xi) = N_{123}(\xi), \quad v_3^{13}(\xi) = N_{133}(\xi)$$
$$v_1^{23}(\xi) = N_{213}(\xi), \quad v_2^{23}(\xi) = N_{223}(\xi) + \xi_3, \quad v_3^{23}(\xi) = N_{233}(\xi)$$
$$v_1^{33}(\xi) = N_{313}(\xi), \quad v_2^{33}(\xi) = N_{323}(\xi), \quad v_3^{33}(\xi) = N_{333}(\xi) + \xi_3$$

$q=3$	$p=1$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5} + 1$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q=3$	$p=2$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5} + 1$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5}$

$q=3$	$p=3$	
Periodicity conditions	in terms of displacements	with inhomogeneities
$v1\|_{\xi_1=-0.5} = v1\|_{\xi_1=0.5}$	$v2\|_{\xi_1=-0.5} = v2\|_{\xi_1=0.5}$	$v3\|_{\xi_1=-0.5} = v3\|_{\xi_1=0.5}$
$v1\|_{\xi_2=-0.5} = v1\|_{\xi_2=0.5}$	$v2\|_{\xi_2=-0.5} = v2\|_{\xi_2=0.5}$	$v3\|_{\xi_2=-0.5} = v3\|_{\xi_2=0.5}$
$v1\|_{\xi_3=-0.5} = v1\|_{\xi_3=0.5}$	$v2\|_{\xi_3=-0.5} = v2\|_{\xi_3=0.5}$	$v3\|_{\xi_3=-0.5} = v3\|_{\xi_3=0.5} + 1$

For solving the nine inhomogeneous periodic problems, which correspond to equations (87) or equations like (35)), the finite element method included in software package ABAQUS should be used.

6.6 Symmetric case

Let $\xi_i = 0$ be a surface of symmetry. Then, as it is known from mechanics,

$$N_j(S_i\xi) = \begin{cases} 0 & \text{if } i = j; \\ N_j(\xi) & \text{if } i \neq j. \end{cases} ; \tag{37}$$

$$\sigma_{ih} n_i \mid_{S_i\xi} = \begin{cases} \sigma_{ih} n_i \mid_{\xi} & \text{if } i = h; \\ 0 & \text{if } i \neq h. \end{cases} \tag{38}$$

For solving the nine equations, which correspond to the equation (87) (since $i_1 = 1, 2 \text{ and } 3$, and for each of them we have 3 equations like (35)), we use the finite element method.
Owing to the symmetry, it is enough to consider $1/8$ of the periodic cell, and, according to 3, Chap.6 in [2], we have only two different types of boundary conditions:

	$i_1 = 1$		
	Equation a) $L_{\xi\xi}(\mathbf{M}^1) = 0$		
$\xi_1 = 0$	$M_1^1 = 0$	$\sigma_2^1 = 0$	$\sigma_3^1 = 0$
$\xi_1 = 0.5$	$M_1^1 = 0.5$	$\sigma_2^1 = 0$	$\sigma_3^1 = 0$
$\xi_2 = 0$	$M_2^1 = 0$	$\sigma_1^2 = 0$	$\sigma_3^2 = 0$
$\xi_2 = 0.5$	$M_2^1 = 0$	$\sigma_1^2 = 0$	$\sigma_3^2 = 0$
$\xi_3 = 0$	$M_3^1 = 0$	$\sigma_1^3 = 0$	$\sigma_2^3 = 0$
$\xi_3 = 0.5$	$M_3^1 = 0$	$\sigma_1^3 = 0$	$\sigma_2^3 = 0$

and the other case:

	$i_1 = 1$		
	Equation b) $L_{\xi\xi}(\mathbf{M}^p) = 0$	$p = 2, 3$	
$\xi_1 = 0$	$\sigma_1^1 = 0$	$M_2^p = 0$	$M_3^p = 0$
$\xi_1 = 0.5$	$\sigma_1^1 = 0$	$M_2^p = \xi_1$	$M_3^p = 0$
$\xi_2 = 0$	$M_1^p = 0$	$\sigma_2^2 = 0$	$M_3^p = 0$
$\xi_2 = 0.5$	$M_1^p = 0$	$\sigma_2^2 = 0$	$M_3^p = 0$
$\xi_3 = 0$	$\sigma_1^3 = 0$	$\sigma_2^3 = 0$	$M_3^p = 0$
$\xi_3 = 0.5$	$\sigma_1^3 = 0$	$\sigma_2^3 = 0$	$M_3^p = 0$

The computation of the effective elasticity tensor is done according to the following equation, for which we use the solution of problem (87) and integrate over the volume of the unit cell:

$$L^H_{i_1 i_2 pq} \;=\; \int_0^1 \int_0^1 \int_0^1 L_{i_1 kpl}(\xi) \frac{\partial}{\partial \xi_k} (N^{i_2 ql}(\xi) \;+\; \xi_{i_2} \delta_{ql})) d\xi. \tag{39}$$

All elements of the elasticity tensor (81 elements) should be computed for an arbitrary cell-geometry.
If the structure of the periodicity cell has a symmetry w.r.t. all cells' main cross-sections and axes of symmetry, this tensor becomes also symmetric and the homogenized material isotropic. Due to the isotropy of the material it is enough to define only two elastic constants:

$$L_{ijhk} = \lambda \delta_{ij} \delta_{hk} + \mu \delta_{ih} \delta_{jk} + \mu \delta_{ik} \delta_{jh}. \tag{40}$$

Choosing adequately those terms, we can determine the averaged values of the Lamé constants. The elements which have been selected are $L^H_{1111} \text{ and } L^H_{1212}$, where:

$$L^H_{1111} = 2\mu^H + \lambda^H \qquad L^H_{2112} = \mu^H \tag{41}$$

Finally, using equation (39) and integrating numerically we obtain the following relations for computations:

$$L^H_{1111} = 2\mu^H + \lambda^H = \sum_e vol_e \cdot \sigma_{11}, \tag{42}$$

$$L^H_{2112} = \mu^H = \sum_e vol_e \cdot \sigma_{21}, \tag{43}$$

where e is the number of elements and $S_{11} \text{ and } S_{12}$ are stress fields obtained by solving (87) with first and second types of boundary conditions, respectively.

Young's Modulus and Poisson's ratio can be calculated with the help of the following relations:

$$\lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

$\mu,\ \lambda$: Lamé's constants,
E : Young's modulus,
ν : Poisson's ratio.

7 Homogenized flow Condition for Ludwik's law

We are going to perform homogenization for the Ludwik's hardening law first.
According to Theorem 4, the homogenized flow condition should be of the same kind as the one of the matrix material, i.e., $\sigma_y^0 = S_0^H + K^H (e_{pl}^0)^n$. It is necessary to check, if both Assumptions 2 and 3 are satisfied for the equivalent stresses given by Ludwik's law. Our idea is to separate the plastic equivalent stress by superposition and perform homogenization for

$$\sigma_{eq}^{\epsilon} := \left(Tr \left(L(\frac{x}{\epsilon}) Du_{el}(x) \right)^2 \right)^{1/2} = S_0(\frac{x}{\epsilon}) \tag{44}$$

and then for

$$\Delta \sigma_{eq}^{\epsilon} = g(\frac{x}{\epsilon}, Du_{pl}^{\epsilon}(x)). \tag{45}$$

The continuous monotone function g from the second expression satisfies all exponential grows conditions from theorem 4, because it is an exponential law of the second invariant of Du_{pl}^{ϵ}, i.e.,

$$g(\frac{x}{\epsilon}, Du_{pl}^{\epsilon}(x)) = K(\frac{x}{\epsilon})(e_{eq}(Du_{pl}^{\epsilon}(x)))^n, \tag{46}$$

where e_{eq} is the bilinear form:

$$e_{eq} = (Tr(Du_{pl}(x))^2)^{1/2}. \tag{47}$$

The only thing that we should remark here, is that the von Mises equivalent stress and strains are the second invariant of the corresponding deviatoric stress or strain tensor, i.e., $\sigma_{eq} = \sqrt{\frac{3}{2} s_{jk}(x) s_{jk}(x)}$, where $s_{ij} = \sigma_{ij} - \delta_{ij} \frac{\sigma_{kk}}{3}$. But since we came into the plastic region, matrix material became incompressible, i.e., $\sigma_{kk} = 0$ and we solve problem for deviators. Therefore, we can conclude that assumptions (iii)-(v) of Assumption 2 will be satisfied for the plasticity with the Ludwik yield rule for the Mises equivalent stresses and the constants c, C will be expressed through parameters S_0 and K. It means, there exists the homogenized macroscopic hardening law. Furthermore, we can prove that the assumption 3 is also satisfied for the equivalent stress given in the form of Ludwik's law: ϵ_{eq}, given by (47) is a homogeneous function of order one of $Du_{pl}(x)$ and moreover g in (46) is a homogeneous function of order n of ϵ_{eq}. Hence, there exists a homogenized macroscopic hardening law and parameters for this macroscopic law can be obtained from the auxiliary periodic cell problems.

7.1 Homogenization of $S(\xi)$

7.1.1 Lower and upper bounds for the homogenized initial flow stress

If $S_0^{\varepsilon}(x)$ is the initial flow stress, then the initial micro-flow-condition can be obtained w.r.t. the algorithm given in Sec. 4. That is, from [1] and [Orlik, 2005], the following two-scale convergence is well-known for pure elastic composites:

$$\underbrace{\sigma_{11}^{\varepsilon}(x)}_{\text{in elastic range}} \overset{\text{(two-scale converges)}}{\rightrightarrows} \sigma_{11}(B(\xi)) \underbrace{\sigma^H}_{1-D},$$

where $B(\xi)$ is a stress concentration factor, $\xi \in Y$, $\xi = \frac{x}{\varepsilon}$. Furthermore, according to Proposition 7, the initial yield condition is

$$\lim_{\epsilon \to 0} ||\sigma_{eq}(\sigma_{11}^{\varepsilon}(x))/S_0(x/\epsilon)||_{C(\Omega)} \leq ||\sigma_{eq}(\sigma_{11}(B(\xi)\sigma_{11}^H)/S_0(\xi)||_{C(Y)} \leq 1,$$

$\forall x \in \Omega,\ \xi \in Y.$

Since the equivalent von Mises stress is the first order homogeneous function of $\sigma_{11}^H := S^H$ (see 2),

$$\lim_{\epsilon\to 0} ||\sigma_{eq}^{\varepsilon}(x)/S_0(x/\epsilon)||_{C(\Omega)} \leq ||\sigma_{eq}(B(\xi))/S_0(\xi)||_{C(Y)} S_0^H \leq 1, \tag{48}$$

$\forall x \in \Omega,\ \xi \in Y.$
How to find $B(\xi)$? It can be found from the solution of the following auxiliary problem on the unit periodicity cell:

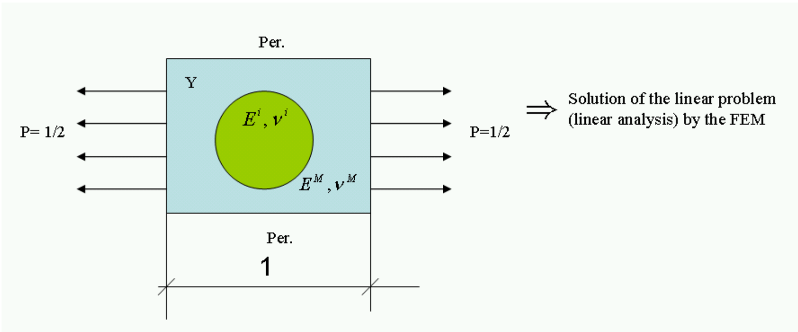


Figure 2 Auxiliary, pure elastic, problem for $B(\xi)$

The stress field from this solution is exactly $B(\xi)$, and $\sigma_{eq}[B(\xi)]$ is the equivalent von Mises stress in this solution. According to Sec. 4

$$S_0^H = \frac{1}{\sup_{\xi\in Y} \frac{\sigma_{eq}[B(\xi)]}{S_0(\xi)}},$$

i.e.,

$$S_0^H = \frac{1}{\max\left(\sup_{\xi\in Y_M} \frac{\sigma_{eq}[B(\xi)]}{S_0^M}, \sup_{\xi\in Y_i} \frac{\sigma_{eq}[B(\xi)]}{S_0^i}\right)},$$

where S_0^i is the yield or ultimative stress for the inclusion's material.
We would like to remark, that stiffer material leads to higher stresses. Nevertheless, is S_0^i so much higher than S_0^M, that we can forecast that

$$S_0^{H\ lower} = \frac{S_0^M}{\sup_{\xi\in Y_M} \sigma_{eq}[B(\xi)]}. \tag{49}$$

For theoretical approval see [Orlik, 2005]. We would like to point out here, that condition (49) characterizes the occurrence of a first plastic point inside the microstructure, therefore we called this value the lower homogenized initial yield limit. We can identify the macroscopic initial yield stress with the state, when the complete matrix material becomes plastic, i.e.,

$$\inf_{\xi\in Y} (\sigma_{eq}(B(\xi))/S_0(\xi)) S_0^H \leq \lim_{\epsilon\to 0} \sigma_{eq}^{\varepsilon}(x)/S_0(x/\epsilon) \leq 1, \qquad \forall x \in \Omega,\ \xi \in Y. \tag{50}$$

To approve this condition we refer to the strong convergence of local equivalent stresses (48), mentioned above and pointed out in [Orlik, 2005]. The strong convergence implies the pointwise one and then we take the lower limit. Finally

$$S_0^{H\ upper} = \frac{S_0^M}{\inf_{\xi\in Y_M} \sigma_{eq}[B(\xi)]}. \tag{51}$$

7.1.2 Homogenized initial flow stress

Another way to find the homogenized parameter S_0 can be taken from [Lipton, 2006],

$$S_0^{H\ average} = \frac{1}{\frac{1}{|Y|}\int_Y \frac{\sigma_{eq}[B(\xi)]}{S_0(\xi)} d\xi},$$

or taking in account that $S_0^i \gg S_0^M$,

$$S_0^{H\ average} = \frac{S_0^M}{\frac{1}{|Y|}\int_Y \chi_M(\xi)\sigma_{eq}[B(\xi)]d\xi} = \frac{S_0^M}{\frac{1}{|Y|}\int_{Y_M} \sigma_{eq}[B(\xi)]d\xi} \qquad (52)$$

where S_0^M is dependent on the parameter S_0 for the matrix material, and $\chi_M,\ \chi_i$ denote characteristic functions of domains in Y occupied by matrix and inclusion material respectively.

7.1.3 Effective initial plastic stress obtained by self-consistent method

If $S_0^\varepsilon(x)$ has not to coincide with a flow stress, then we can find it by a self-consistent strategy, as an averaged over the microstructure stress corresponding to the same elastic strain, which occurs under the uni-axial stress $\sigma_{11} := S_0^M$ in the homogenous sample from the pure matrix material. I.e.,

$$S_0^{H\ eff.\ extension} = \frac{\hat{E}^H}{E_M} S_0^M \qquad (53)$$

or

$$S_0^{H\ eff.\ hydrostatic} = \frac{\hat{k}^H}{k_M} S_0^M. \qquad (54)$$

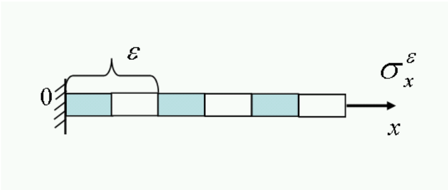
where E_M, k_m are the Young's and bulk moduli of the matrix material and $\hat{E}^H$, $\hat{k}^H$ are the homogenized Young's and bulk moduli of the composite.
Remark 11 *Values obtained later on in this report by formulas (67) and (68) should be close to those obtained by formula (52). This statement requires an additional explanation. The auxiliary problem for calculation of the effective elastic constants in Sec. 6 is the elasticity problem for the composite unit cell with boundary conditions given in terms of periodic displacement components with only one additional unit displacement in the direction defining kind of the 1-Dimensional experiment.*
The auxiliary problem for the stress concentration 2 considered in this section, considers the same structure and physics of the unit cell and the same 1-D experiment, but with the boundary conditions in the stress terms. Since the problem is linear, solutions of these two problems coincide within some scaling parameter and one boundary condition can be easily reformulated in terms of the other one by the formula given in Sec.5, [Orlik, 2005]: $\sigma_{ij}^0(x,\xi) = B_{ijkl}(\xi)\hat{\sigma}_{kl}(x)$, *where* $B_{ijkl}(\xi) = L_{ijh\gamma}(\xi)\frac{\partial}{\partial \xi_\gamma}(N_{hpq}(\xi) + \xi_q\delta_{hp})L_{pqkl}^{H-1}$ *is the stress concentration tensor and* $\frac{1}{|Y|}\int_Y B_{ijkl}(\xi)d\xi = \delta_{ij}\delta_{kl}$.

7.2 Homogenization of $K(\xi)$.

It is obvious and is also mentioned in ["Lecture notes" of Suquet], that the whole deformation process can be separated into 3 stages: in the first one both, inclusions and matrix, are object to only elastic deformations. In the second (which is actually very short and will be neglected in our work), the matrix material starts already to be object to plastic deformations, but they are still smaller than the elastic deformations of the inclusions. In the third stage the plastic deformation of the matrix material is much larger than the elastic deformation of the inclusions, and then the lust one can be neglected so that the inclusions are assumed rigid in this stage. We would like to consider the third stage in this subsection.

7.2.1 1-D Homogenization based on rheological models and self-consistency

(a) **Maxwell model** (Reuss)
Consider a bar of the length l represented by two materials, which are switched in an ε-periodic series, as it is shown in figure 3.



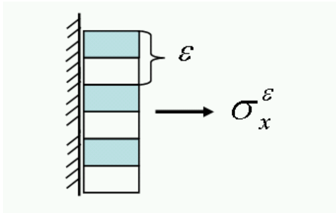


Figure 3 1-D Maxwell model Figure 4 1-D Voigt model

$$\sigma^\varepsilon(x) = \sigma(x,\xi) = \sigma^H(x) \quad \forall x,\xi;\; e^\varepsilon(x) = \sum_{l/\varepsilon} e(\xi) = \frac{l}{\varepsilon}(e_m + e_i)$$

$$e_{pl}(\xi) = K(\xi)^{-1/n}(\sigma^H - S_0^H)^{1/n}, \text{ then} \tag{55}$$

$$e_{pl}^H = \left(\frac{1}{|Y|}\int_Y K^{-1/n}(\xi)d\xi\right)(\sigma^H - S_0^H)^{1/n},$$

i.e.,

$$K^H = \left(\frac{1}{|Y|}\int_Y K^{-1/n}(\xi)d\xi\right)^{-n} \tag{56}$$

(b) **Voigt model**
Let us consider now material composed as in figure 4 with the period of structure ε.

$$\begin{cases} \sigma_x^\varepsilon = \sum_{l/\varepsilon}(\sigma(\xi)) = \frac{l}{\varepsilon}(\sigma_m + \sigma_i) \\ e_{pl\ x}^\varepsilon = e_{m\ pl\ x} = e_{i\ pl\ x} = e_{pl\ x}(x,\xi) = e_{pl\ x}^H \end{cases}$$

$$\sigma(\xi) = S_0^H + K(\xi)(e_{pl}^H)^n, \text{ i.e. } \begin{cases} \sigma_m = S_0^H + K_m(e_{pl}^H)^n \\ \sigma_i = S_0^H + K_m(e_{pl}^H)^n \end{cases}, \ 0 \le \xi \le 1.$$

That is

$$\sigma^H = \frac{1}{|Y|}\int_Y \sigma(\xi)d\xi = S_0^H + K^H(e_{pl}^H)^n,$$

where

$$K^H = \frac{1}{|Y|}\int_Y K(\xi)d\xi \tag{57}$$

Remark 12 *According to Sec. 1.6 of [Suquet, "Lecture Notes"], the first bound (with Maxwell representation of composite) is called Reuss bound and is a lower one for K^H while the Voigt bound is an upper bound.*

7.2.2 3-D Homogenization for $K(\xi)$

(a) Let us restrict ourselves only on the plastic stress range, i.e. only stress values after occurrence of the plastic deformations reduced by S_0:

$$\Delta\sigma_Y^\varepsilon(\overline{e}_{pl}^\varepsilon) := \sigma_Y^\varepsilon - S_0^M.$$

W.r.t. [1], [4] and our Sec. 2,

$$\Delta\sigma_{pl}^\varepsilon(e_{pl}) \quad \overset{\text{(two-scale converges)}}{\rightrightarrows} \quad \Delta\sigma_{pl}(\xi, \nabla_\xi(\xi + N(\xi)) \cdot e_{pl}^{(0)}(x))$$

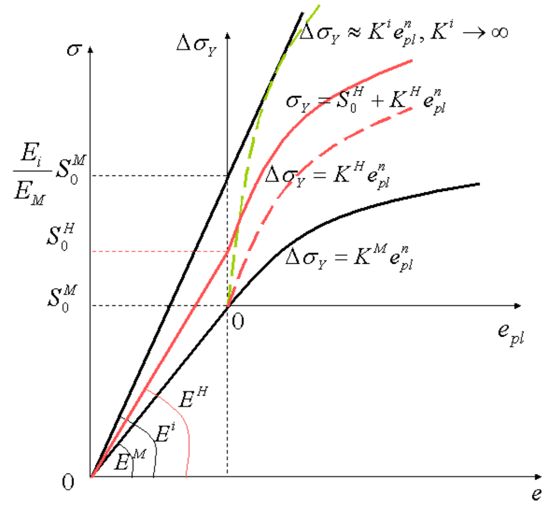


Figure 5 Homogenization scheme for elasto-plasticity illustrated by 1-D σ-e diagram

Now, if we recall that $\sigma_{pl\ eq} = \Lambda(\sigma) = \sigma_Y$ and apply Proposition 7, we obtain

$$\begin{aligned} &\lim_{\epsilon\to 0} \sup_{x\in\Omega} |\Delta\sigma_Y^{\varepsilon}(\overline{e}_{pl}) - \Delta\sigma_Y(\xi, \nabla_\xi(\xi + N(\xi))|_{\xi=x/\epsilon} \cdot \overline{e}_{pl}^{(0)}(x))| \\ = &\lim_{\epsilon\to 0} \sup_{x\in\Omega} |\Delta\sigma_Y^{\varepsilon}(\overline{e}_{pl}) - \Delta\sigma_Y(\xi, 1 + \overline{e}_{pl}^{1}(\xi))|_{\xi=x/\epsilon}(\overline{e}_{pl}^{(0)}(x))^n|. \end{aligned}$$

That is,

$$\lim_{\epsilon\to 0} \sup_{x\in\Omega} |\Delta\sigma_Y^{\varepsilon}(\overline{e}_{pl})| \le \sup_{x\in\Omega} \sup_{\xi\in Y} |\Delta\sigma_Y(\xi, \overline{e}_{pl}^{cell}(\xi))(\overline{e}_{pl}^{0}(x))^n|. \quad (58)$$

Or we can find the two-scale limit for the equivalent stress $\sigma_{pl\ eq}$ directly. Since the equivalent deviatoric stress is an exponential function of the trace of the squared deformation deviatoric tensor and therefore satisfies the condition of monotonicity and exponential growth, the Mitty lemma can be applied to prove its two-scale convergence by monotonicity. That is

$$|\Delta\sigma_Y^{\varepsilon}(\overline{e}_{pl})| \overset{\text{weak } \varepsilon\to 0}{\longrightarrow} \left(\frac{1}{|Y|}\int_{|Y|} \Delta\sigma_Y(\xi, \overline{e}_{pl}^{\text{cell}})d\xi\right)(\overline{e}_{pl}^{0}(x))^n. \quad (59)$$

The convergence and limit given by (58) is completely approved by [Orlik, 2005], while (59) requires an additional rigorous proof.
For the Ludwik law

$$\Delta\sigma_Y(\xi, 1 + \overline{e}_{pl}^{1}(\xi)) = K(\xi)(1 + \overline{e}_{pl}^{1}(\xi))^n,$$

$$\overline{e}_{pl}^{1}(\xi) \text{ is a 1-periodic function and } (1 + \overline{e}_{pl}^{1}(\xi)) = \overline{e}_{pl}^{\text{cell}}(\xi)$$

Consequently,

$$K^H = \begin{cases} \frac{1}{|Y|}\int_Y \Delta\sigma_{eq}(\xi, \overline{e}_{pl}^{\text{cell}})d\xi, \text{ or} \\ \sup_{\xi\in Y_M} |\Delta\sigma_{eq}(\xi, \overline{e}_{pl}^{cell}(\xi))|. \end{cases} \quad (60)$$

Let us now remark that the inclusion material is pure elastic and that there exists no explicite K^i. We can approximate inclusion material behaviour corresponding to the plastic hardening in matrix by the same exponential law as for matrix (green line on figure above). Furthermore, we refer to p.194 of the book [3], where it is proved that in case of rigid or very stiff inclusions the integral in the last expression should be taken only over the matrix volume and for the cell boundary conditions we should take either

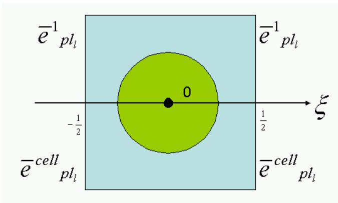
$$U^{cell} = U^0 + \text{rigid displacement}$$

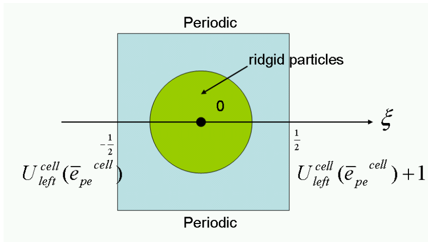
at the interface between matrix and inclusion materials, or

$$e^{cell}(\xi)|_{\xi \in Y_M} = e^0.$$

This statement require also more explanations.

Boundary conditions for the cell problem:





$$\Delta\sigma_Y^{\text{cell}} = K(\xi)(\overline{e}_{pl}^{\text{cell}}(\xi))^n,$$

$$U^{\text{cell}} = \int_{|Y|} \overline{e}_{pl}^{\text{cell}}(\xi)d\xi = \xi + N(\xi)$$

$N(\xi)$ is periodic function.

(b) Let us apply the stress S_0^H to the cell. Then the flow rule for the **auxiliary problem** will be:

$$\sigma_y^{\text{cell}}(\xi) = S_0^H + K(\xi)(\overline{e}_{pl}^{\text{cell}}(\xi))^n$$

The auxiliary problem can be then solved with homogenized elastic properties, i.e.

$$e^{\text{cell}}(\xi) = e_{pl}^{\text{cell}}(\xi) + \frac{\sigma^{\text{cell}}(\xi)}{E^H}$$

Remark 13 *The auxiliary problem can be solved with the flow rule:*

$$\sigma_y^{\text{cell}}(\xi) = S_0(\xi) + K(\xi)(\overline{e}_{pl}^{\text{cell}}(\xi))^n.$$

In this case $K^H = \frac{1}{|Y|}\int_Y (\sigma_{pl}^{\text{cell}}(\xi) - S_0(\xi))d\xi.$

Further assumption:

$$\sigma_y|_{\xi=-\frac{1}{2}} = \sigma_y|_{\xi=\frac{1}{2}} = S_0^H + \Delta\sigma_y^H,$$

then

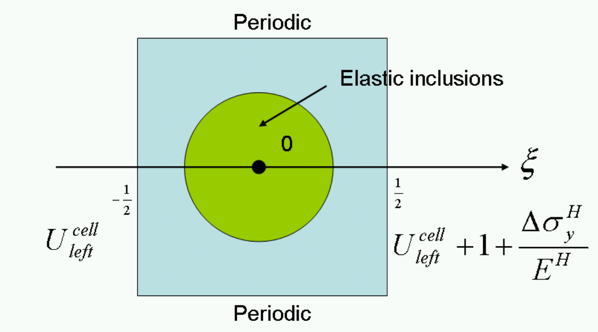
$$U^{\text{cell}}(\xi) = \xi + N(\xi) + \frac{S^H + \Delta\sigma_y^H}{E^H}\xi,$$

$N(\xi)$ is periodic.

8 Homogenized flow Condition for Hocket-Sherby's law

The Hocket-Sherby's plastic law is written in the following way:

$$\sigma^\epsilon(x) = b(\frac{x}{\epsilon}) - (b(\frac{x}{\epsilon}) - a(\frac{x}{\epsilon}))\exp(-m(\frac{x}{\epsilon})(e_{pl}^\epsilon(x))^n).$$



It can be checked, that the Hocket-Sherby condition satisfies Assumption 2 but not Assumption 3. That means that there exists a limiting homogenized hardening law, but the problems on the periodicity cell and on the macroscopic body can not be separated. Nevertheless, we will try to separate them by some trick, namely perform homogenization for the flow limit $b(\xi)$ exactly as for the term $S(\xi)$ from the previous section and then define

$$\Delta\sigma(x,\xi) := -(b(\xi)-a(\xi))\exp(-m(\xi)\cdot(e_{pl}^n(x,\xi))),\ \ \xi=\frac{x}{\epsilon}.$$

Further, we can consider $\ln(\Delta\sigma^\epsilon(\xi))$, which reduces this law to the Ludwik's law:

$$\ln(\Delta\sigma(x,\xi)) := \ln(a(\xi)-b(\xi)) - m(\xi)\cdot(e_{pl}^n(x,\xi)).$$

Now we can formally use for parameters $\ln(a(\xi)-b(\xi))$ the same homogenization procedure as for $S(\xi)$ from the Ludwik's model, and for $-m(\xi)$ - the same as for $K(\xi)$ from Ludwik's law.

9 Calculation of equivalent parameters for a specific composite material

In this section we consider an elasto-plastic composite consisting of a metallic matrix and silica, pure elastic particles. Both, inclusion and matrix materials are isotropic and their elastic properties are known. The matrix material possesses one of the following hardening plastic relations

$$\sigma_{eq} = \begin{cases} \sigma_{pl}(x) = S_0 + K(\overline{e}_{pl})^n(x) & \text{Ludwik} \\ \sigma_{pl}(x) = b-(b-a)\exp(-m\cdot(\overline{e}_{pl})^n(x)) & \text{Hocket-Sherby} \end{cases}$$

where σ_{eq} and $\overline{e}_{pl}$ are equivalent Mises stress and plastic strains and $S_0,\ K,\ n$ and $a,\ b,\ m,\ n$ are known parameter sets.
For the solution of the cell problems we used the commercial finite element (FE) software ABAQUS with its postprocessor, integration of the corresponding stress components over the periodicity cell, was performed with the help of own python scripts.

Given for elastic part and Ludwik's law:

Volume fraction of inclusion: $c = 0.10$.
Number of elements: 57898
Number of strings with stresses in 4 integr. points/el-t: 231592.

	Inclusions characteristics	Matrix characteristics
E	$4.8 \cdot 10^5.0$	$4.5 \cdot 10^4$
ν	0.20	0.32
μ	$2.0 \cdot 10^5$	17045.45455
k	$2.67 \cdot 10^5$	41666.67
S_0		33.47
K		305.2
n		0.1726

Table 1 Elastic and Ludwig's constants

Numerically obtained averaged elastic and plastic parameters:

For calculation of the homogenized elastic coefficients we solved 2 boundary value problems (BVP) for a pure elastic composite with $E_m,\ \nu_M,\ E_i,\ \nu_i$.
The coefficient $\lambda_{eq} + 2\mu_{eq}$ was obtained from the solution of the BVP in the $1/8$ unit cell with periodic boundary conditions and extension $U_1|_{x_1=0} = 0.0,\ U_1|_{x_1=0.5} = 0.5$. Since the elasticity problem is linear, the solution depends linearly on the boundary data. Therefor, in order to stay in the infinitesimal strain theory, we applied $U_1|_{x_1=0.5} = 0.005$ instead of $U_1|_{x_1=0.5} = 0.5$, solved the problem and then multiplied the solution by the factor 100.
The coefficient μ_{eq} was obtained from the solution of BVP in the $1/8$ unit cell with periodic boundary conditions and shear $U_2|_{x_1=0} = 0.0,\ U_2|_{x_1=0.5} = 0.005$. The solution again previously was multiplied by 100.
$\lambda_{eq} + 2\mu_{eq} = 72219.48, \mu_{eq} = 20103.09, k_{eq} = 45415.35, E_{eq} = 52554.82, \nu_{eq} = 0.31$ Matrix volume $= 0.112574$, volume of $1/8$ of the periodic cell $= 0,125$.

$$S_{0\ eq\ yield} = 12.73,$$
$$S_{0\ eq\ aver\ ext} = \frac{E_{eq}}{E_M} S_{0\ M} = 39.09,$$
$$S_{0\ eq\ aver\ hydr} = \frac{k_{eq}}{k_M} S_{0\ M} = 36.48$$

For calculation of the homogenized coefficient k we solved (BVP) for almost pure plastic matrix with constitutive law $\sigma = 0.03 + ke^n_{pl}$ and rigid or very stiff ($E_i = 48 \cdot 10^9$) inclusions with B.C. unit tension. Since the pure plastic problem $\sigma = ke^n_{pl}$ is homogeneous of the order n, the solution also depends homogeneously of the order n on the boundary data. Therefore, to avoid numerical problems with abaqus, we again applied $U_1|_{x_1=0.5} = 0.05$ instead of $U_1|_{x_1=0.5} = 0.5$, solved the problem and then multiplied the solution by 100.
$K_{aver} = 396.98, K_{aver\ matr.} = 244.87, K_{max\ M} = 329.76$.

For Hocket-Sherby plastic model:

For the matrix material we considered the following parameter set: $a = 82.3$, $b = 165$, $m = 2561$, $n = 1.216$.

Numerical results for the averaged elastic and plastic parameters:

$m^H_{rigud\ particl.} = 3216.41, m^H_{stiff\ particl.} = 5142.46, m^H_{max\ M} = 4425.62, \ln(b_{eq} - a_{eq})_{ext} = 5.16, \ln(b_{eq} - a_{eq})_{hydr} = 4.81, b_{eq\ ext} = 192.70, b_{eq\ hydr} = 179.84, a_{eq\ ext} = 19.15, a_{eq\ hydr} = 56.81$

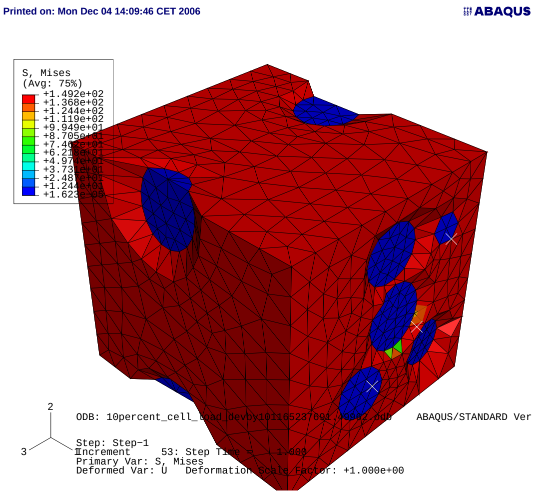


Figure 6 Mises stress in the periodicity cell. Solution of problem with pure plastic matrix, $\sigma = k\left(\frac{e_{pl}}{10}\right)^n$, and rigid inclusions. B.C. $U_1|_{x_1=0.5} = 0.05$

10 Mechanical averaging

10.1 Averaging of elastic properties

We consider the composite with the following elastic coefficients for inclusion and matrix materials:

$$E_i = 480000.0,\ \nu_i = 0.20,\ E_m = 45000.0,\ \nu_m = 0.32,$$

where index i is related to inclusion and m - to matrix materials, and the inclusion volume fraction is $c = 0.1$. Let us consider some relations and simple averaging formulas well-known from mechanical literature. First of all, let us recall the following relations for two-constant sets, Lame-constants or Young modulus and Poisson's ratio, describing elastic properties of isotropic materials:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}, \qquad k = \lambda + \frac{2}{3}\mu$$

And the back formulas are:

$$E = \frac{9k\mu}{3k+\mu}, \qquad \nu = \frac{3k-2\mu}{2(3k+\mu)}$$

Hashin's composite sphere model for elastic coefficients

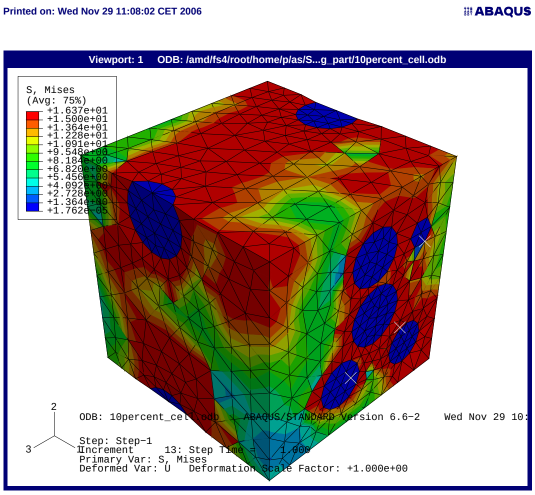


Figure 7 Mises stress in the periodicity cell. Solution of problem with pure plastic matrix, $\sigma = m\left(\frac{e_{pl}}{100}\right)^n$, and rigid inclusions

Suppose, the composite material consists of composite spheres of different sizes, where we call a composite sphere a spherical particle from the inclusion material, of radius a, embedded into the spherical matrix phase, of radius b. The composite spheres model assumes that for each composite sphere the ratio a/b is supposed to be a constant, independent of the absolute particle size (Christensen [6], Ch.2).

Let us cut out a single composite sphere. It is subjected to a hydrostatic traction p on its outer boundary. We consider additionally an equivalent sphere of homogeneous material with unknown elastic constants, which is subjected to the same outer traction p. Then we define the continuity conditions for radial displacement and normal radial stresses on the interface $r = a$ between inclusion and matrix materials. Furthermore, on the boundary $r = b$, the radial displacements should take the same values for the composite sphere and the sphere of the equivalent homogeneous material, radial normal stresses should also be identical and be equal to the outer traction p.

Substituting a general solution of the single Lamé equation of the spherically symmetric elasticity problem (Christensen [6]) to the inclusion, the matrix, and the equivalent sphere, avoiding the singularity at $r = 0$, and satisfying the continuity conditions, we get

$$k_{eq} = k_m + \frac{c(k_i - k_m)}{1 + (1-c)[(k_i - k_m)/(k_m + 4/3\mu_m)]}, \tag{61}$$

where c, k and μ are the volume fraction of the particles, bulk and shear moduli, and the indices i, m, eq are related to the inclusion, the matrix and the equivalent homogeneous sphere respectively.

Lower and upper bounds

From the potential energy minima principle, the following boundaries for the bulk modulus

$$\frac{c}{1+(1-c)[(k_i-k_m)/(k_m+4/3\mu_m)]} \leq \frac{k_{eq}-k_m}{k_i-k_m} \leq \frac{c}{1+(1-c)[(k_i-k_m)/(k_m+4/3\mu_i)]}, \tag{62}$$

and for the shear modulus

$$\frac{c}{1+(1-c)[(\mu_i-\mu_m)/(\mu_m+\mu_l)]} \leq \frac{\mu_{eq}-\mu_m}{\mu_i-\mu_m} \leq \frac{c}{1+(1-c)[(\mu_i-\mu_m)/(\mu_m+\mu_u)]} \tag{63}$$

are known, where

$$\mu_l = \frac{3}{2}\left(\frac{1}{\mu_m}+\frac{10}{9k_m+8\mu_m}\right)^{-1}, \qquad \mu_u = \frac{3}{2}\left(\frac{1}{\mu_i}+\frac{10}{9k_i+8\mu_i}\right)^{-1} \tag{64}$$

For the considered low inclusion concentration we stay closer to the lower boundary.
Calculations of elastic coefficients for our special case
We found out that $\mu_i = 2.0\cdot 10^5$, $k_i = 2.67\cdot 10^5$, $\mu_m = 17045.45$, $k_m = 41666.67$. And

$$47095.28 \leq k_{eq} \leq 55247.41, \quad 20344.38 \leq \mu_{eq} \leq 27448.65$$

$$E_{eq}^{Hashin} = 53350.91, \quad \nu_{eq}^{Hashin} = 0.31.$$

10.2 Effective parameters for plastic Ludwik and Hocket-Sherby laws obtained by self-consistent methods

We shell consider two forms of the plastic laws for matrix material, namely Ludwik's law:

$$\sigma = S_0 + Ke_{pl}^n, \tag{65}$$

and the Hocket-Sherby law:

$$\sigma = b-(b-a)\exp(-me_{pl}^n). \tag{66}$$

Let us consider the Ludwik law first and then show that the Hocket-Sherby law can be also transformed to the Ludwik law and, hence, the same homogenization algorithm can be applied for it too.
Ludwik's plasticity
If S_0 has not to coincide with a flow stress, then we can find it by a self-consistent strategy, as an average over the microstructure stress corresponding to the same elastic strain, which occurs under the uni-axial stress $\sigma_{11} := S_0^M$ in the homogenous sample from the pure matrix material. I.e.,

$$S_{0\ eq\ ext} = \frac{\hat{E}_{eq}}{E_M} S_0^M \tag{67}$$

or

$$S_{0\ eq\ hydr} = \frac{\hat{k}_{eq}}{k_M} S_0^M. \tag{68}$$

where E_M, k_m are the Young's modulus and bulk modulus of the matrix's material and $\hat{E}_{eq}$, $\hat{k}_{eq}$ are the homogenized Young's and bulk moduli of the composite.
Further, we use the averaging formulas of Suquet (see Annex) for power-law plasticity for matrix materials and rigid inclusions:

$$K_{eq} = K^M \frac{1+\frac{d}{2}c}{(1-c)^n}, \tag{69}$$

or

$$K_{eq} = K^M \frac{(1 + \frac{d}{2}c)^{\frac{n+1}{2}}}{(1-c)^n}. \quad (70)$$

Here d is dimension of the problem, i.e. $d = 3$ in our case.

$$n_{eq} := n.$$

Calculations for Ludwik's model
After performing the calculations for Ludwik's law (9) of matrix material with constants $S_0 = 33.47,\ K = 305.2,\ n = 0.1726$, we obtain:

$$S_{0\ eq\ ext} = 39.68, \quad S_{0\ eq\ hydr} = 37.83.$$

Furthermore, using the averaging formulas of Suquet (69) and (70),

$$K_{eq} = 357.42, \qquad \text{or} \qquad K_{eq} = 337.34.$$

Hocket-Sherby's plasticity
Now, consider the Hocket-Sherby plastic law (66). We will perform the averaging for the flow limit b exactly as for the term S from the Ludwik's law and then define

$$-\Delta\sigma := (b-a)\exp(-m \cdot e_{pl}^n).$$

Further, we consider $\ln(\Delta\sigma)$, which reduces this law to the Ludwik's law:

$$\ln(-\Delta\sigma) := \ln(b-a) - m \cdot e_{pl}^n.$$

Now we can formally use for parameters $\ln(a-b)$ the same homogenization procedure as for S from the Ludwik's model, and for $-m$ - the same as for K from Ludwik's law.
Calculations for Hocket-Sherby's model
For the matrix material we have: $a = 82.3$, $b = 165$, $m = 2561$, $n = 1.216$. Using formulas (67) and (68),

$$b_{eq\ ext} = 195.62, \quad b_{eq\ hydr} = 186.50.$$

$$\ln(b-a)_{eq\ ext} = 5.23, \quad \ln(b-a)_{eq\ hydr} = 4.99.$$

That is,

$$a_{eq\ ext} = 7.97, \quad a_{eq\ hydr} = 39.49$$

Furthermore, using the averaging formulas of Suquet (69) and (70),

$$m_{eq} = 3347.72, \qquad \text{or} \qquad m_{eq} = 3398.63.$$

We also performed mechanical averaging with volume fraction $c = 12\%$. The calculated averaged characteristics are given in tables 10.2-10.2 below.

In Fig. 8, the effective Hook's law obtained by asymptotic homogenization is graphically compared with the result of the mechanical averaging. It can be seen that averaged lines are captured between those for matrix and inclusion materials. Figures 9 and 10 demonstrate the same for parameters K and m from the Ludwik and Hocket-Sherby laws respectively. Finally, we compared effective constitutive laws obtained by homogenization and averaging with experimental data obtained by BMW. The comparence demonstrated a good approximation of the numerical curves by our model except of the region around the small plastic strains, where they are comparable with elastic deformations of inclusion material. The reason is that the averaged curve is glued from parts calculated for pure elasticity and those for plastic matrix and rigid inclusion. For a correct approximation of the experiment it might be helpful to consider for the phase of small plastic deformations an additional curve in-between, which should contain K^H calculated by averaging of the cell solution for the problem with pure plastic particles and matrix both modeled by Ludwik's law $\Delta\sigma = K \cdot (e_{pl})^n$, where K for inclusions should be rather high and taken in such a way that it approximates the line $\Delta\sigma = S_{0\ M} + E_i \cdot e_{pl}$ (green curve on the Fig. 5).

	Inclusions characteristics	Matrix characteristics	Equivalent characteristics, mechanical averaging	Equivalent characteristics, assymtotic homogenization
E	$4.8 \cdot 10^5.0$	$4.5 \cdot 10^4$	53350.91	52554.82
ν	0.20	0.32	0.31	0.31
μ	$2.0 \cdot 10^5$	17045.45	20344.38	20103.09
k	$2.67 \cdot 10^5$	41666.67	47095.28	45415.35

Table 2 Elasticity (volume fraction $c = 10\%$)

	Matrix characteristics	Equivalent characteristics, mechanical averaging	Equivalent characteristics, assymtotic homogenization
S_0	33.47	$S_{0_{ext}} = 39.68$ $S_{0_{hyd}} = 37.83$	$S_{0_{ext}} = 39.09$ $S_{0_{hyd}} = 36.48$
K	305.2	$K_1 = 357.42$ $K_2 = 337.34$	$K = 396.98$ $K_{matr} = 244.87$ $K_{maxM} = 329.76$
n	0.1726	0.1726	0.1726

Table 3 Ludwik plasticity (volume fraction $c = 10\%$)

	Matrix characteristics	Equivalent characteristics, mechanical averaging	Equivalent characteristics, assymtotic homogenization
b	165	$b_{ext} = 195.62$ $b_{hyd} = 186.50$	$b_{ext} = 192.70$ $b_{hyd} = 179.84$
$\ln(b-a)$	4.42	$\ln(b-a)_{ext} = 5.23$ $\ln(b-a)_{hyd} = 4.99$	$\ln(b-a)_{ext} = 5.16$ $\ln(b-a)_{hyd} = 4.81$
a	82.3	$a_{ext} = 7.97$ $a_{hyd} = 39.49$	$a_{ext} = 19.15$ $a_{hyd} = 56.81$
m	2561	$m_1 = 3347.71$ $m_2 = 3398.63$	$m_{\text{rigud partic.}} = 3216.41$ $m_{\text{aver. with stiff partic.}} = 5142.46$
n	1.216	1.216	1.216

Table 4 Hocket-Sherby plasticity (volume fraction $c = 10\%$)

	Inclusions characteristics	Matrix characteristics	Equivalent characteristics, mechanical averaging
E	$4.8 \cdot 10^5$	$4.5 \cdot 10^4$	55201.78
ν	0.20	0.32	0.31
μ	$2.0 \cdot 10^5$	17045.45	21077.61
k	$2.67 \cdot 10^5$	41666.67	48292.72

Table 5 Elasticity (volume fraction 12%)

	Matrix characteristics	Equivalent characteristics, mechanical averaging
S_0	33.47	$S_{0_{ext}} = 41.06$ $S_{0_{hydr}} = 38.79$
K	305.2	$K_1 = 368.17$ $K_2 = 343.80$
n	0.1726	0.1726

Table 6 Ludwik plasticity (volume fraction 12%)

	Matrix characteristics	Equivalent characteristics, mechanical averaging
b	165	$b_{ext} = 202.41$ $b_{hyd} = 191.24$
$\ln(b-a)$	4.42	$\ln(b-a)_{ext} = 5.42$ $\ln(b-a)_{hyd} = 5.12$
a	82.3	$a_{ext} = -22.61$ $a_{hyd} = 24.35$
m	2561	$m_1 = 3530.21$ $m_2 = 3593.88$
n	1.216	1.216

Table 7 Hocket-Sherby plasticity (volume fraction 12%)

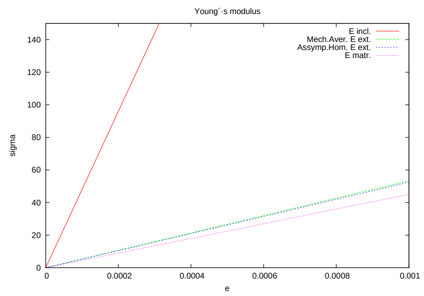
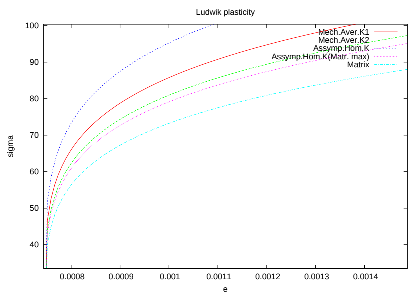


Figure 8 Averaging of elastic properties



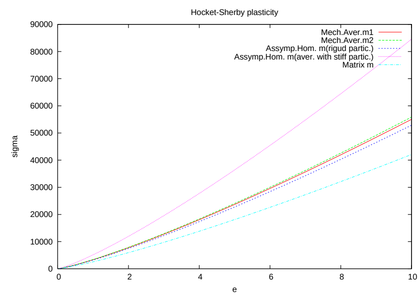


Figure 9 Ludwik plasticity $\Delta\sigma = K(e_{pl})^{0.1726}$ Figure 10 Ludwik plasticity $\Delta\sigma = m(e_{pl})^{1.216}$

11 Appendix: EFFECTIVE PROPERTIES OF NONLINEAR COMPOSITES of P. Suquet

We denote by $c^{(r)} = \frac{1}{|V|}\int_v x^{(r)}(x)dx,$ the volume fraction of the inclusions.

A class of nonlinear composites

In most of this study the constitutive law of the individual constituents can be put as

$$\sigma_m = \kappa tr(\varepsilon), s_{ij} = 2\mu_s(\varepsilon_{eq})e_{ij}, \tag{71}$$

where σ_m and ε_m are the first invariants of the stress and strain, s and e are the stress and strain deviators,

$$\sigma_m = \frac{1}{3}tr(\boldsymbol{\sigma}), \qquad \varepsilon_m = \frac{1}{3}tr(\boldsymbol{\varepsilon}),$$
$$s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}, \qquad e_{ij} = \varepsilon_{ij} - \varepsilon_m\delta_{ij},$$

σ_{eq} and ε_{eq} are the Von Mises equivalent stress and equivalent strain

$$\sigma_{eq} = (\frac{2}{3}s_{ij}s_{ij})^{\frac{1}{2}}, \qquad \varepsilon_{eq} = (\frac{2}{3}e_{ij}e_{ij})^{\frac{1}{2}}.$$

and μ_s is the secant shear modulus

$$\mu_s = \frac{\sigma_{eq}}{3\varepsilon_{eq}}.$$

The response of the phases is therefore assumed to be linear for purely hydrostatic loadings and nonlinear in shear. This assumption is made to simplify the developments but is by no means a restriction of the methods described below. The constitutive law (71) derives from the potential

$$\varphi(\varepsilon) = \frac{1}{2}\kappa tr(\varepsilon)^2 + f(\varepsilon_{eq}^2), \qquad f(x) = \int_0^{x^{\frac{1}{2}}} 3\mu_s(e)ede \tag{72}$$

Note that

$$\mu_s(\varepsilon_{eq}) = \frac{2}{3}f'(\varepsilon_{eq}^2), \tag{73}$$

where f' denotes the derivative of f with respect to its scalar argument x.

Secant formulation

The constitutive law (71) can be alternatively written in a compact form

$$\boldsymbol{\sigma} = \mathbb{L}_s(\varepsilon) : \varepsilon, \tag{74}$$

with

$$\mathbb{L}_s(\varepsilon) = 3\kappa\mathbb{J} + 2\mu_s(\varepsilon_{eq})\mathbb{K}, \tag{75}$$

$\mathbb{L}_s$ is the secant stiffness tenor. The relation (75) shows its decomposition on the two projectors $\mathbb{J}$ and $\mathbb{K}$:

$$\left.\begin{array}{c} J_{ijkh} = \frac{1}{3}\delta_{ij}\delta_{kh}, \quad K_{ijkh} = \frac{1}{2}(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) - J_{ijkh}, \\ \mathbb{J}:\mathbb{J} = \mathbb{J}, \quad \mathbb{K}:\mathbb{K} = \mathbb{K}, \quad \mathbb{K}:\mathbb{J} = \mathbb{O}, \quad \mathbb{J}+\mathbb{K} = \mathbb{I}, \end{array}\right\} \tag{76}$$

where $\mathbb{I}$ is the identity in the space of symmetric fourth order tensors.

Power-law materials

We shall often make reference to the specific class of $incompressible\ power-law\ materials$ for the purpose of comparing predictive schemes. The two potentials φ and ψ read in this case

$$\varphi(\varepsilon) = \frac{\sigma_0\varepsilon_0}{m+1}(\frac{\varepsilon_{eq}}{\varepsilon_0})^{m+1} \quad \text{when} \quad tr\varepsilon = 0, \quad \varphi(\varepsilon) = +\infty \quad \text{otherwise}, \tag{77}$$

and

$$\psi(\boldsymbol{\sigma}) = \frac{\sigma_0\varepsilon_0}{n+1}(\frac{\sigma_{eq}}{\sigma_0})^{n+1}. \tag{78}$$

The constitutive law reads

$$\boldsymbol{\sigma} = \sigma_m \boldsymbol{I} + s, \quad \frac{s}{\sigma_{eq}} = \frac{2}{3}\frac{\boldsymbol{e}}{\varepsilon_{eq}}, \quad \frac{\sigma_{eq}}{\sigma_0} = (\frac{\varepsilon_{eq}}{\varepsilon_0})^m. \tag{79}$$

The exponents m and n are related by $m = \frac{1}{n}$.

- When $m = n = 1$ the above potentials describe the behavior of a linear incompressible material with shear modulus $\mu = \frac{\sigma_0}{3\varepsilon_0}$.
- When $m = 0, n = +\infty$, the potentials describe the behavior of a rigid-plastic material with flow stress σ_0.

In most applications considered here the exponent m varies between 0 and 1. The secant tensor $\mathbb{L}_s$ read

$$\mathbb{L}_s = +\infty\mathbb{J} + 2\mu_s(\varepsilon_{eq}) = \frac{1}{3}\frac{\sigma_0}{\varepsilon_0}^{m-1}, \tag{80}$$

and

$$\mathbb{L}_t = +\infty\mathbb{J} + 2\mu_t(\varepsilon_{eq})(\mathbb{F} + m\mathbb{E}), \quad \mu_t(\varepsilon_{eq}) = \mu_s(\varepsilon_{eq}), \quad \lambda = m\mu_t \tag{81}$$

Remark 1: A straightforward calculation shows that the tensor $\frac{1}{m}\mathbb{L}_t$ can also be used as a secant tensor to define the constitutive law (79).

Effective potentials

In the case of power-law materials the Reuss and Voigt bounds read

$$\frac{\sigma_0^R\varepsilon_0}{m+1}(\frac{\varepsilon_{eq}}{\varepsilon_0})^{m+1} \le \Phi(\boldsymbol{E}) \le \frac{\sigma_0^V\varepsilon_0}{m+1}(\frac{\varepsilon_{eq}}{\varepsilon_0})^{m+1},$$
$$\sigma_0^R = <\sigma_0^{-n}>^{-m}, \sigma_0^V = <\sigma_0>.$$

(In our case $n = \frac{1}{m}$, and these are bounds for our K from Ludwik's law.)
When $m = 0$ (rigid-plastic materials) the Reuss bound simplifies to

$$\sigma_o^R = \inf_{r=1,\ldots,N} \sigma_0^{(r)}.$$

Hashin-Shtrikman and Willis bounds

$$\left.\begin{array}{c}\mathbb{L}^{HS-}=3k^{HS-}-\mathbb{J}+2\mu^{HS-}\mathbb{K},\\ k^{HS-}=k^{(2)}+c^{(1)}\frac{k^{(1)}-k^{(2)}}{1+c^{(2)}\frac{k^{(1)}-k^{(2)}}{kp}},\quad \mu^{HS-}=\mu(2)+c^{(1)}\frac{\mu^{(1)}-\mu(2)}{1+c^{(2)}\frac{\mu^{(1)}-\mu(2)}{\mu p}}.\end{array}\right\} \tag{82}$$

Classical secant method. Power-law materials

The composites considered in this section are two-phase power-law materials with overall isotropy. The individual constituents are governed by the relations (71). They have the same exponents n and m and the same strain ε_0 but different flow stresses σ_0. The Hashin-Shtrikman bound (82) is used. Detail calculations are presented here for 3-dimensional isotropic composites. Most of the results are also valid for unidirectional composites (or can be easily extended).
The nonlinear problem

$$\left.\begin{array}{c}E_{eq}^{(1)}=a_{eq}^{(1)}E_{eq},\quad E_{eq}^{(2)}=a_{eq}^{(2)}E_{eq},\\ \mu^{(1)}=\mu_s^{(1)}(E_{eq}^{(1)}),\quad \mu^{(2)}=\mu_s^{(2)}(E_{eq}^{(2)}),\\ \mu^{\text{hom}}=\mu^{\text{hom}}(\mu^{(1)},\mu^{(2)},c^{(1)}).\end{array}\right\} \tag{83}$$

can be further simplified by using the homogeneity of degree m of the constitutive law and the explicit expression of the secant modulus μ_s given in (ï£¡80). The nonlinear equations reduce to the scalar equation

$$\frac{E_{eq}^{(1)}}{E_{eq}^{(2)}}=\frac{1}{1+(\frac{\mu^{(1)}}{\mu^{(2)}-1})},\quad \frac{\mu^{(1)}}{\mu^{(2)}}=\frac{\sigma_0^{(1)}}{\sigma_0^{(2)}}(\frac{E_{eq}^{(1)}}{E_{eq}^{(2)}})^{m-1}, \tag{84}$$

where $\beta=\frac{2}{2+d}$. This equation can be rearranged into

$$(1-\beta)\frac{E_{eq}^{(1)}}{E_{eq}^{(2)}}+\beta\frac{\sigma_0^{(1)}}{\sigma_0^{(2)}}(\frac{E_{eq}^{(1)}}{E_{eq}^{(2)}})^m=1. \tag{85}$$

Rigid inclusions

When phase 1 is rigid, the nonlinear equation (84) and the average condition

$$c^{(1)}E_{eq}^{(1)}+c^{(2)}E_{eq}^{(2)}=E_{eq}$$

can be combined to show that

$$\lim_{\sigma_0^{(1)}\to+\infty} E_{eq}^{(1)} = 0, \quad \lim_{\sigma_0^{(1)}\to+\infty} E_{eq}^{(2)} = \frac{E_{eq}}{c^{(2)}}, \quad \lim_{\sigma_0^{(1)}\to+\infty} \sigma_0^{(1)}(E_{eq}^{(1)})^m = \frac{1}{\beta}\sigma_0^{(2)}(E_{eq}^{(2)})^m.$$

The deviator of the overall stress is

$$\mathbb{S} = c^{(1)}\mathbb{S}^{(1)} + c^{(2)}\mathbb{S}^{(2)},$$

where, using

$$\hat{\varepsilon} = \hat{\varepsilon}^{(1)} = \hat{\varepsilon}^{(2)}, \quad \hat{\varepsilon}^{(r)} = \frac{\varepsilon^{(r)}}{E_{eq}^{(r)}}, \quad \hat{\varepsilon} = \frac{\varepsilon}{E_{eq}} \tag{86}$$

$$\mathbb{S}^{(1)} = \lim_{\sigma_0^{(1)}\to+\infty} \frac{2}{3}\sigma_0^{(1)}(\frac{E_{eq}^{(1)}}{\varepsilon_0})^m\hat{\varepsilon}, \quad \mathbb{S}^{(2)} = \frac{2}{3}\sigma_0^{(2)}(\frac{E_{eq}^{(2)}}{\varepsilon_0})^m\hat{\varepsilon}.$$

Finally

$$\mathbb{S} = \frac{2}{3}\frac{\sigma_0^{\mathsf{hom}}}{\varepsilon_0}(\frac{E_{eq}}{\varepsilon_0})^{m-1}\varepsilon,$$

with

$$\sigma_0^{\mathsf{hom}} = \sigma_0^{(2)}\frac{(1+\frac{d}{2}c^{(1)})}{c^{(2)^m}}. \tag{87}$$

Modified method. Two-phase power-law materials

Nonlinear equation

In this section we apply the modified method to incompressible power-law materials with the same exponent. The nonlinear problem

$$\mathbb{L}^{(r)} = \mathbb{L}_s^{(r)}(\overline{\overline{\varepsilon}}_{eq}^{(r)}), \quad \overline{\overline{\varepsilon}}_{eq}^{(r)} = (\frac{1}{3c^{(r)}}\boldsymbol{E} : \frac{\partial\mathbb{L}^{\mathsf{hom}}}{\partial\mu^{(r)}}(k,\mu) : \boldsymbol{E})^{\frac{1}{2}}. \tag{88}$$

reduces to a scalar equation which takes the following from (except for porous materials where the problem is even simpler)

$$\frac{\overline{\overline{\varepsilon}}_{eq}^{(1)}}{\overline{\overline{\varepsilon}}_{eq}^{(2)}} = \left(\frac{c^{(2)}}{c^{(1)}}\frac{\frac{\partial\mu^{\mathsf{hom}}}{\partial\mu^{(1)}}}{\frac{\partial\mu^{\mathsf{hom}}}{\partial\mu^{(2)}}}\right)^{\frac{1}{2}}, \quad \frac{\mu^{(1)}}{\mu^{(2)}} = \frac{\sigma_0^{(1)}}{\sigma_0^{(2)}}\left(\frac{\overline{\overline{\varepsilon}}_{eq}^{(1)}}{\overline{\overline{\varepsilon}}_{eq}^{(2)}}\right)^{m-1}. \tag{89}$$

When the effective shear modulus μ^{hom} of the comparison solid is estimated by the Hashin-Shtrikman lower bound, this equation can be rearranged to give

$$\left(\frac{\overline{\overline{\varepsilon}}_{eq}^{(1)}}{\overline{\overline{\varepsilon}}_{eq}^{(2)}}\right)^2 = \left(1+\beta\left(\frac{\mu^{(1)}}{\mu^{(2)}}-1\right)\right)^2 + \beta(1-\beta)c^{(1)}\left(\frac{\mu^{(1)}}{\mu^{(2)}}-1\right)^2. \tag{90}$$

Remark 2: The volume fractions of the phases enter the above nonlinear equation : the ratio between the average strain in the two phases predicted by the modified secant method depends on the volume fraction of the phases.

Equation (90) cannot be solved in a closed form, except when the phases are rigid-plastic ($m = 0$) (see below) and is solved numerically.

Rigid inclusions

When phase 1 is rigid $\overline{\overline{\varepsilon}}_{eq}^{(1)} = E_{eq}^{(1)} = 0$. The energy equation

$$\mu^{\text{hom}} = c^{(1)}\mu^{(1)}\left(\frac{\overline{\overline{\varepsilon}}_{eq}^{(1)}}{E_{eq}}\right)^2 + c^{(2)}\mu^{(2)}\left(\frac{\overline{\overline{\varepsilon}}_{eq}^{(2)}}{E_{eq}}\right)^2,$$

and the nonlinear equation (89) show that

$$\lim_{\sigma_0^{(1)}\to+\infty} \overline{\overline{\varepsilon}}_{eq}^{(2)} = \frac{\left(1+\frac{1-\beta}{\beta}c^{(1)}\right)^{\frac{1}{2}}}{c^{(2)}}E_{eq}, \quad \lim_{\sigma_0^{(1)}\to+\infty} \sigma_0^{(1)}\,\overline{\overline{\varepsilon}}_{eq}^{(1)m} = \frac{\sigma_0^{(2)}\,\overline{\overline{\varepsilon}}_{eq}^{(2)m}}{\beta\left(1+\frac{1-\beta}{\beta}c^{(1)}\right)^{\frac{1}{2}}}.$$

Finally the effective constitutive law reads

$$\mathbb{S} = \frac{2}{3}\frac{\sigma_0^{\text{hom}}}{\varepsilon_0}\left(\frac{E_{eq}}{\varepsilon_0}\right)^{m-1}\varepsilon,$$

with

$$\sigma_0^{\text{hom}} = \sigma_0^{(2)}\frac{\left(1+\frac{d}{2}c^{(1)}\right)^{\frac{m+1}{2}}}{c^{(2)m}}. \tag{91}$$

Incremental method. Incompressible power-law materials

Rigid inclusions.
The overall flow stress is:

$$\sigma_0^{\text{hom}} = \sigma_0^{(2)}\frac{c^{(2)} + \frac{m+1}{2mC}c^{(1)}}{c^{(2)m}}. \tag{92}$$

Variational procedures

Nonlinear materials containing voids or rigid inclusions

The composites considered in this section consist of a nonlinear matrix with voids and (or) rigid inclusions.

$$\sigma_0^{\text{hom}} = \sigma_0^{(2)} \frac{\left(1 + \frac{d}{2} c^{(1)}\right)^{\frac{m+1}{2}}}{c^{(2)^m}}.$$

An incremental model based on the variational procedure

A simple incremental model which applies to metal matrix composites reinforced by $particles$ can be proposed by neglecting the deformation of the particles. First recall the construction of constitutive laws such as

$$\left.\begin{array}{c} \varepsilon(u) = \varepsilon^e + \varepsilon^p, \quad \boldsymbol{\varepsilon}^e = \mathbb{M}_0^{(2)} : \boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}^p = \frac{3}{2}\dot{p}\frac{\boldsymbol{s}}{\sigma_{eq}}, \\ \dot{p} = \frac{3}{2}\frac{1}{h(p)}(s : \dot{s})^+ \text{ when } \sigma_{eq} = \sigma_Y(p), \quad \dot{p} = 0 \text{ when } \sigma_{eq} < \sigma_Y(p). \end{array}\right\} \tag{93}$$

by two thermodynamic potentials. The model is defined by three state variables, $\varepsilon, \varepsilon^p$ and p in this specific situation and two thermodynamic potentials, the Helmholtz free energy $\mathcal{H}$ and the dissipation potential $\mathcal{D}$ of the material. The free energy is split into two terms, the elastic energy, recoverable by unloading the material, and the stored energy:

$$\mathcal{H}(\varepsilon, \varepsilon^p, p) = \underbrace{\frac{1}{2}(\varepsilon - \varepsilon^p) : \mathbb{L}^{(2)} : (\varepsilon - \varepsilon^p)}_{\text{Elastic energy}} + \underbrace{\varphi(p)}_{\text{Stored energy}}, \varphi(p) = \int_0^p \sigma_Y(p) dp.$$

The thermodynamic forces are defined by the $state\ laws$ as

$$\boldsymbol{\sigma} = -\frac{\partial \mathcal{H}}{\partial \varepsilon^p} = \mathbb{L}_0^{(2)} : \varepsilon^e, \quad R = -\frac{\partial \mathcal{H}}{\partial p} = -\sigma_Y(p), \tag{94}$$

The power dissipated in the irreversible processes taking place in the material is

$$\mathcal{D} = \boldsymbol{\sigma} : \varepsilon^p + R\dot{p}.$$

The evolution of the internal variables describing irreversible phenomena are given by the $complementary\ laws$:
$Yield\ funktion$

$$\mathcal{F}(\boldsymbol{\sigma}, R) = \sigma_{eq} + R,$$

$flow\ rule$

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \mathcal{F}}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}, R), \dot{p} = \dot{\lambda}\frac{\partial \mathcal{F}}{\partial R}(\boldsymbol{\sigma}, R).$$

The approximate model for the overall behavior of MMC's is also based on the description of dissipative constitutive laws by two overall potentials, the overall free energy of the composite and its dissipation potential. The approximations introduced to estimate these potentials are based on the observation that one can distinguish three regimes in the deformation of the composites:

1. $Elastic\ regime$: the plastic strains in the matrix are smaller than the elastic strains.
2. $Elastic - plastic\ regime$: the elastic and plastic strains in the matrix are of the same order.
3. $Plastic\ regime$: the elastic strains in the matrix and in the inclusions are smaller than the plastic strains in the matrix.

The elastic regime can be handled by any linear theory appropriate for the microstructure of the composite providing the overall stiffness $\mathbb{L}^{\text{hom}}$ of the composite and, consequently, its overall elastic energy. The elastic-plastic regime is neglected in the present model. The plastic regime is handled by considering the particles as rigid inclusions in a plastic matrix with hardening. The overall stored energy of the composite is estimated by considering that the matrix is a nonlinear elastic material with a strain energy density $\varphi(\varepsilon)$ (deformation theory of plasticity with elastic deformations neglected). When the composite has overall isotropy, the overall stored energy potential can be estimated by :

$$\Phi(P) = c^{(2)}\varphi\left(\left(\frac{\mu^{\text{hom}}}{c^{(2)}\mu_0}\right)^{\frac{1}{2}} P\right), \tag{95}$$

where μ^{hom} is the effective shear modulus of a linear composite made up of an incompressible matrix with shear modulus μ^0 containing rigid inclusions. The expression (95) makes use of the fact that the overall strain reduces to to the overall plastic strain in the plastic regime $\boldsymbol{E} \simeq \boldsymbol{E}^p$ and $E^p_{eq} = P$. The two terms entering the Helmholtz free energy of the composite, the elastic energy, and the stored energy are estimated

$$\mathcal{H}^{\text{hom}}(\boldsymbol{E}, \boldsymbol{E}^p, P) = \frac{1}{2}(\boldsymbol{E} - \boldsymbol{E}^p) : \mathbb{L}^{\text{hom}} : (\boldsymbol{E} - \boldsymbol{E}^p) + \Phi(P).$$

The state laws read

$$\Sigma = -\frac{\partial \mathcal{H}^{\text{hom}}}{\partial \boldsymbol{E}^p} = \boldsymbol{L}^{\text{hom}} : (\boldsymbol{E} - \boldsymbol{E}^p), R^{\text{hom}} = -\frac{\partial \mathcal{H}^{\text{hom}}}{\partial P} = -\sigma_Y^{\text{hom}}(P),$$

$$\sigma_Y^{\text{hom}}(P) = c^{(2)} C \sigma_Y(CP), C = \left(\frac{\mu^{\text{hom}}}{c^{(2)}\mu_0}\right)^{\frac{1}{2}}. \tag{96}$$

The overall yield function $\mathcal{F}^{\text{hom}}$ is the usual Von Mises criterion:

$$\mathcal{F}^{\text{hom}}(\Sigma, R^{\text{hom}}) = \Sigma_{eq} + R^{\text{hom}},$$

and the normality law at the overall scale reads

$$\dot{\Sigma}^p = \dot{\Lambda}\frac{\partial \mathcal{F}^{\text{hom}}}{\partial \Sigma}(\Sigma, R^{\text{hom}}), \quad \dot{P} = \dot{\Lambda}\frac{\partial \mathcal{F}^{\text{hom}}}{\partial R}(\Sigma, R^{\text{hom}}).$$

Finally, the proposed effective constitutive law of the composite is

$$\left.\begin{array}{c} \boldsymbol{E} = \boldsymbol{E}^e + \boldsymbol{E}^p, \quad \boldsymbol{E}^e = \mathbb{M}^{\text{hom}} : \Sigma, \\ \dot{\Sigma}^p = \frac{3}{2}\dot{P}\frac{\mathbb{S}}{\Sigma_{eq}} \ \text{ when } \ \Sigma_{eq} = \sigma_Y^{\text{hom}}(P), \quad \dot{\Sigma}^p = 0 \ \text{ when } \ \Sigma_{eq} < \sigma_Y^{\text{hom}}(P). \end{array}\right\} \tag{97}$$

This approximate model requires only one constant (C in(96)) which is deduced from a linear theory.
Example 14 *the above model was implemented for material with*

$$\sigma_Y(p) = S_0 + Ap^{\alpha}.$$

The homogenized overall yield stress for the case of rigid particles in the incompressible matrix is:

$$\sigma_Y^{hom}(P) = \frac{5c^{(1)} + 2c^{(2)}}{2}\sigma_Y^{(2)}\left(\frac{5c^{(1)} + 2c^{(2)}}{2c^{(2)}}P\right).$$

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