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On the general equivalence of the Fried parameter and coherence radius for non-Kolmogorov and oceanic turbulence

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Abstract: This article aims to discuss the general equivalence between two main metrics used in laser beam propagation through random media and imaging: the Fried parameter and the coherence radius. In particular, we show that their relationship deduced for Kolmogorov and non-Kolmogorov turbulence stays valid for other media as long as their spectral power laws are equal.

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1. Introduction

Fried parameter, r_0 is among the most used metrics to quantify the effects of optical turbulence on imaging, while the coherence radius, ρ_0 is more commonly used in problems involving laser beam propagation. The former is based on asymptotic behavior of the resolution metric [1] while the latter is deduced from the wave structure function [2]. If the Kolmogorov power spectrum of the fluctuations of the index of refraction is used to describe turbulence, it is well known that r_0 and ρ_0 are related by the simple equation $r_0 = 2.1 \cdot \rho_0$ [2]. However, it is very useful to extend this equivalence also to cases of different power spectra such as the non-Kolmogorov power spectra [3,4] or the oceanic turbulence power spectrum [5].

2. General coherence ratio

Let us to find how r_0 and ρ_0 are related if the following power law-dependent non-Kolmogorov power spectrum is used [3]:

$$\Phi_n(\kappa, \alpha) = A(\alpha) \cdot \tilde{C}_n^2 \cdot \kappa^{-\alpha}, \qquad 3<\alpha<4 \tag{1}$$

where $A(\alpha)\frac{\Gamma(\alpha-1)}{4\pi^2}\cos\left(\alpha\frac{\pi}{2}\right)$, $\vec{\kappa} \equiv (\kappa_x, \kappa_y, \kappa_z)$ is the spatial wavenumber vector, α is the power law, $\tilde{C}_n^2 = \beta \cdot C_n^2$ is the generalized structure parameter with units $[m^{3-\alpha}]$, β is a dimensional constant with units $[m^{11/3-\alpha}]$ and symbol $\Gamma(x)$ denotes the Gamma function. Equation (1) is valid only in the inertial sub-range, $2\pi/L_0<<\kappa<<2\pi/l_0$, where l_0 is the inner scale and L_0 is the outer scale of turbulence. Kolmogorov power law assumes value $\alpha = 11/3$, and then the generalized structure parameter reduces to the structure parameter C_n^2 with units $[m^{-2/3}]$.

In this article, we aim to find the general equivalence $r_0(\alpha) = c_0(\alpha) \cdot \rho_0(\alpha)$ between the Fried parameter and coherence radius valid also for non-Kolmogorov turbulence or oceanic turbulence and that reduces to the Kolmogorov case, $r_0 = 2.1 \cdot \rho_0$ when $\alpha = 11/3$.

The coherence radius, $\rho_0(\alpha)$ is defined by the 1/e point of the modulus of the complex degree of coherence DOC, which is given by

$$DOC(\rho, \alpha) = \exp\left[-\frac{1}{2}D(\rho, \alpha)\right] \tag{2}$$

where $D(\rho, \alpha)$ is the wave structure function and ρ is the separation distance.

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The wave structure function $D(\rho, \alpha)$ is the sum of the phase structure function, $D_S(\rho, \alpha)$ and the log-amplitude structure function, $D_\chi(\rho, \alpha)$. We make the usual assumption that $D_S(\rho, \alpha)$ is the dominant term:

$$D(\rho, \alpha) = D_S(\rho, \alpha) + D_\chi(\rho, \alpha) \cong D_S(\rho, \alpha) \tag{3}$$

Note that $D_\chi(\rho, \alpha) = 0$ if the geometric optics approximation is invoked (see pag. 195 in Ref. [2]), therefore Eq. (3) is valid for separation distance $\rho >> \sqrt{L/k}$.

The 1/e point of the modulus of the complex degree of coherence DOC is reached by imposing $D_S(\rho_0, \alpha) = 2$, which allows us to extract $\rho_0(\alpha)$. If non-Kolmogorov spectrum, (1), is used, we can express the phase structure function as

$$D_S(\rho, \alpha) = 2 \cdot \left[\frac{\rho}{\rho_0(\alpha)}\right]^{\alpha-2} \tag{4}$$

For a given horizontal propagation path of length L, a given structure constant $\tilde{C}_n^2$ and wavelength λ, the plane wave phase structure function can obtained from Eq. (8) in Ref. [4] setting inner scale equal to zero and outer scale equal to infinite,

$$D_S(\rho, \alpha) = -4\pi^2 A(\alpha) \cdot k^2 \cdot L \cdot \tilde{C}_n^2 \cdot \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)} \cdot \left(\frac{\rho}{2}\right)^{\alpha-2} = K_{\text{atm}}(\alpha) \cdot \rho^{\alpha-2} \tag{5}$$

where

$$K_{\text{atm}}(\alpha) = -4\pi^2 A(\alpha) \cdot k^2 \cdot L \cdot \tilde{C}_n^2 \cdot \frac{\Gamma(1-\alpha/2)}{\Gamma(\alpha/2)} \cdot 2^{2-\alpha} \tag{6}$$

and $k = 2\pi/\lambda$ is the wavenumber.

Imposing $D_S(\rho_0, \alpha) = 2$ leads to the coherence radius

$$\rho_0(\alpha) = \left[-\frac{2^\alpha \cdot \Gamma(\alpha/2)}{8\pi^2 A(\alpha) \cdot k^2 \cdot L \cdot \tilde{C}_n^2 \cdot \Gamma(1-\alpha/2)}\right]^{\frac{1}{\alpha-2}} \tag{7}$$

The Fried parameter $r_0(\alpha)$ can be related to the phase structure function by

$$D_S(r, \alpha) = c_1(\alpha) \cdot \left[\frac{r}{r_0(\alpha)}\right]^{\alpha-2} \tag{8}$$

where $c_1(\alpha = 11/3) = 6.88$ for the Kolmogorov case.

Stribling et al. [6] found the expression for $c_1(\alpha)$. He followed the approach of Fried [1], who set the constant c_1 based on the asymptotic behavior of the resolution metric $R/R_{\max} = SR \cdot (D/r_0)^2$, where SR is the Strehl ratio, D is the diameter of the receiver aperture, R is the resolution and $R_{\max}$ is the maximum resolution. The resolution is defined as

$$R = 2\pi \int_0^\infty \omega \cdot H(\omega) d\omega$$

where $H(\omega)$ is the long exposure optical transfer function (OTF) of an incoherent imaging system in the presence of turbulence. Also, the maximum resolution is defined as the resolution of an imaging system in turbulence with an infinite aperture diameter, $R_{\max} = \lim_{D\to\infty} R$.

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For completeness, we show here the main steps Stribling et al. [6] followed to obtain $c_1(\alpha)$. First, we observe that [6]

$$\frac{R}{R_{\max}} = \frac{32}{\pi}\int_0^{D/r_0} \frac{1}{2}\left(\cos^{-1}\left(\frac{r_0\cdot x}{D}\right) - \left(\frac{r_0\cdot x}{D}\right)\sqrt{1-\left(\frac{r_0\cdot x}{D}\right)^2}\right)\exp\left[-\frac{1}{2}c_1(\alpha)\cdot x^{\alpha-2}\right]xdx \quad (9)$$

then, as suggested by Fried [1], let us increase the diameter D to infinity:

$$\lim_{D\to\infty}\frac{R}{R_{\max}} = 8\cdot\int_0^{\infty}\exp\left[-\frac{1}{2}c_1(\alpha)\cdot x^{\alpha-2}\right]xdx \quad (10)$$

By making the change of variable $t = \frac{c_1}{2}x^{\alpha-2}$ and using the formula $\Gamma(x) = s^x\cdot\int_0^{\infty}\exp(-s\cdot t)\cdot t^{x-1}dt$, Eq. (10) reduces to

$$\lim_{D\to\infty}\frac{R}{R_{\max}} = \frac{8}{\alpha-2}\cdot\Gamma\left(\frac{2}{\alpha-2}\right)\cdot\left[\frac{2}{c_1(\alpha)}\right]^{\frac{2}{\alpha-2}} \quad (11)$$

Because $\lim_{D\to\infty}\frac{R}{R_{\max}} = 1$, we can extract

$$c_1(\alpha) = 2\left[\frac{8}{\alpha-2}\cdot\Gamma\left(\frac{2}{\alpha-2}\right)\right]^{\frac{\alpha-2}{2}} \quad (12)$$

Equation (12) was already obtained by Stribling [6]. Note that $c_1(\alpha = 11/3) = 6.88$ (as previously stated) is the well-known value for Kolmogorov turbulence.

Now, by replacing $r = c_0(\alpha)\cdot\rho$ in Eq. (8) we deduce that

$$D_S(r,\alpha) = D_S[c_0(\alpha)\cdot\rho,\alpha] = c_1(\alpha)\cdot\left(\frac{c_0(\alpha)\cdot\rho}{c_0(\alpha)\cdot\rho_0(\alpha)}\right)^{\alpha-2} = c_1(\alpha)\cdot\frac{D_S(\rho,\alpha)}{2} \quad (13)$$

After that, using Eqs. (5) and (13), we deduce that $D_S(r,\alpha) = D_S[c_0(\alpha)\cdot\rho,\alpha]$ is equivalent to the equation

$$K_{\text{atm}}(\alpha)\cdot[c_0(\alpha)\cdot\rho]^{\alpha-2} = K_{\text{atm}}(\alpha)\cdot\frac{c_1(\alpha)}{2}\cdot\rho^{\alpha-2} \quad (14)$$

Using Eq. (12) and Eq. (14) the equivalent coherence ratio for non-Kolmogorov turbulence is

$$c_0(\alpha) = \left[\frac{c_1(\alpha)}{2}\right]^{\frac{1}{\alpha-2}} = \left[\frac{8}{\alpha-2}\cdot\Gamma\left(\frac{2}{\alpha-2}\right)\right]^{\frac{1}{2}} \quad (15)$$

We plot in Fig. 1 the equivalent coherence ratio $c_0(\alpha)$ as function of the power law α. We deduce that the coherence ratio can assume values in the range $2 \le c_0(\alpha) \le 2.83$ depending on the α value.

Note that the constant $K_{\text{atm}}(\alpha)$ does not play any role in the equivalence. It means that for a different turbulence model, let's say oceanic turbulence, $c_0(\alpha)$ remains the same if the oceanic wave structure function has the same power law as the atmospheric case. Also, note that, for $\alpha = 11/3$ Eq. (15) reduces to the well-known value for the Kolmogorov turbulence case, $c_0(\alpha = 11/3) = 2.1$.

The requirement of the same power law between two wave structure functions is quite strong but not prohibitive. In fact, even very different random media, in the inertial range, may show the same power laws (the atmosphere and the ocean for example show approximately the same power law 11/3). Let us describe this possible situation.

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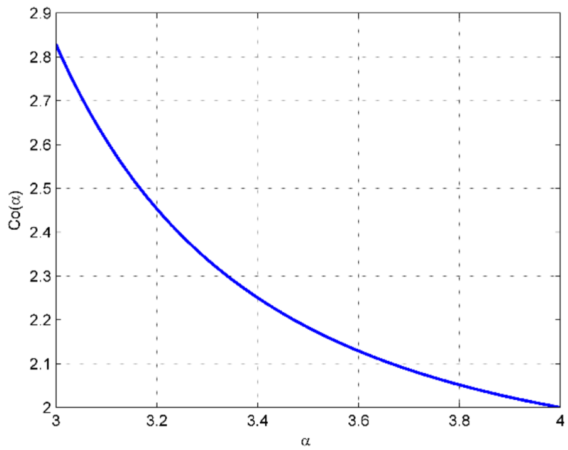

Fig. 1. Equivalence of the coherence ratio $c_0(\alpha)$ as function of the power law α.

3. Equivalent coherence ratio for oceanic turbulence

The fluctuations of the index of refraction in the ocean can be modeled by the following power spectrum [5]

$$\Phi_n(\kappa) = 0.388 \cdot 10^{-8} \cdot \varepsilon^{-1/3} \cdot \kappa^{-11/3} \cdot [1 + 2.35(\kappa\eta)^{2/3}] \cdot \frac{X_T}{w^2} \cdot (w^2 \cdot e^{-A_T \cdot \delta} + e^{-A_S \cdot \delta} - 2 \cdot e^{-A_{TS} \cdot \delta}) \tag{16}$$

Here, κ is the spatial wavenumber, ε is the rate of dissipation of kinetic energy per unit mass of fluid, X_T is the rate of dissipation of mean-squared temperature, η is the Kolmogorov microscale, w is a unit-less parameter specifying the ratio of temperature to salinity contributions to the refractive index spectrum, $-5 \leq w<0$, where $w = -5$ means temperature dominated turbulence and $w \to 0$ salinity dominated. Note that power spectrum (16) is not valid for the case $w = 0$, see Appendix B of Ref. [5]. In addition:

$$A_T = 1.863 \cdot 10^{-2}; A_S = 1.9 \cdot 10^{-4}; A_{TS} = 9.41 \cdot 10^{-3} \ \delta = 8.284 \cdot (\kappa\eta)^{\frac{4}{3}} + 12.978 \cdot (\kappa\eta)^2.$$

The wave structure function of a plane wave in turbulence is [2]

$$D(\rho, L) = 8\pi^2 \cdot k^2 \cdot L \cdot \int_0^\infty \kappa \cdot \Phi_n(\kappa) \cdot e^{-\frac{\Delta L}{k} \cdot \xi^2 \cdot \kappa^2} \cdot [1 - J_0(\kappa\rho)] d\kappa \tag{17}$$

Equation (17) was solved in Ref. [5] for oceanic turbulence by using the software Mathematica. We solved it in closed form and found

$$\begin{aligned} D(\rho, L) = {} & 8\pi^2 \cdot k^2 \cdot L \cdot \chi_T \cdot \varepsilon^{-\frac{1}{3}} \cdot 0.388 \cdot 10^{-8} \cdot w^{-2} \cdot \left[w^2 \cdot \left(S_{T,-11/3} + 2.35\eta^{\frac{2}{3}} S_{T,3}\right)\right. \\ & \left. + \left(S_{S,-11/3} + 2.35\eta^{\frac{2}{3}} S_{S,3}\right) - 2w\left(S_{TS,-11/3} + 2.35\eta^{\frac{2}{3}} S_{TS,3}\right)\right] \end{aligned} \tag{18}$$

where

$$S_{i=[T,S,TS],-11/3} = (A_i \cdot a_4)^{\frac{5}{6} - \frac{2}{3}n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \frac{(A_i \cdot a_3)^n}{n!} \Gamma\left(\frac{2}{3}n - \frac{5}{6}\right) \cdot \left[1 - {}_1F_1\left(\frac{2}{3}n - \frac{5}{6}; 1; -\frac{\rho^2}{4 \cdot a_4 \cdot A_i}\right)\right] \tag{19}$$

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and

$$S_{i=[T,S,TS],-3} = (A_i \cdot a_4)^{\frac{1}{2}-\frac{2}{3}n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \frac{(A_i \cdot a_3 \cdot)^n}{n!} \Gamma\left(\frac{2}{3}n - \frac{1}{2}\right) \cdot \left[1 - {}_1F_1\left(\frac{2}{3}n - \frac{1}{2}; 1; -\frac{\rho^2}{4 \cdot a_4 \cdot A_i}\right)\right] \tag{20}$$

Here, $a_3 = 8.284 \cdot \eta^{4/3}$ and $a_4 = 12.978 \cdot \eta^2$ and ${}_1F_1$ is the hypergeometric function

Retaining only the term n = 0 of the series, using the approximation [2,5]

$${}_1F_1(a; c; -z) \cong \frac{\Gamma(c)}{\Gamma(c-a)} \cdot z^{-a}, \mathrm{Re}(z) >> 1 \tag{21}$$

and under the condition $\rho >> \eta$ we obtain

$$D_S(\rho)_{\text{ocean}} = K_{\text{ocean}} \cdot \rho^{\frac{5}{3}} \tag{22}$$

where

$$K_{ocean} = -3.6303 \cdot 10^{-7} \cdot \frac{\Gamma(-5/6)}{\Gamma(11/6)} \cdot 2^{-\frac{8}{3}} \cdot k^2 \cdot L \cdot \varepsilon^{-\frac{1}{3}} \cdot \frac{\chi_T}{w^2} \cdot (w^2 + 1 - 2w) \tag{23}$$

Closed-form Eq. (22) is equivalent to Eq. (10) in Ref. [5] where the authors used Mathematica.

Note that the condition $\rho >> \eta$ is satisfied if separation distance $\rho >> 0.3cm$ (Kolmogorov scale is $\eta \cong 10^{-3}m$) which is quite realistic. Please see Ref. [5] where authors use this value in Figs. 1, 2 and 3.

We also note that the exponent in Eq. (22) is $\alpha - 2 = 5/3$ which is equivalent to the Kolmogorov case, $\alpha = 11/3$, therefore $c_0(\alpha = 11/3) = 2.1$ and we deduce that, although under approximation (retaining only the term n = 0 of the series and under the condition $\rho >> \eta$) the equivalence between the Fried parameter and coherence radius for oceanic turbulence is the same as for Kolmogorov turbulence, $r_0 = 2.1 \cdot \rho_0$. Similar conclusion can be made for light propagation in other media if the wave structure function shows the 5/3 power law.

4. Summary

In general, for a random medium having the wave structure function that can be expressed in the form

$$D_S(\rho)_{\text{medium}} = K_{\text{medium}} \cdot \rho^{\alpha-2}, \qquad 3 < \alpha < 4 \tag{24}$$

the equivalent coherence ratio $c_0(\alpha)$ will be always given by Eq. (15) whatever the medium is.

The equivalent coherence ratio, $c_0(\alpha)$, allows one to relate the coherence radius to the Fried parameter for any random medium with a wave structure function having the form of Eq. (22). This result is useful for several applications involving laser beam propagation through the atmosphere, the ocean, and even in tissue [7]. In optical imaging, for example, it is very convenient to express the phase variances as a function of the ratio D/r_0, therefore the equivalent coherence ratio $c_0(\alpha)$ can allow to do that if the coherence radius $\rho_0(\alpha)$ is known.

We report here, for completeness, another useful relationship which is not explicitly shown in Ref. [6], to relate the coherence radius of a spherical wave to the coherence radius of a plane wave in non-Kolmogorov turbulence:

$$\frac{\rho_{0\text{ spherical}}}{\rho_{0\text{ plane}}} = \frac{r_{0\text{ spherical}}}{r_{0\text{ plane}}} = (\alpha - 1)^{\frac{1}{\alpha-2}} \tag{25}$$

Note that Eq. (25) reduces to $\rho_{0\text{ spherical}} = 1.8 \cdot \rho_{0\text{ plane}}$ for the Kolmogorov case. It is shown in Ref. [2] that the coherence radius of a Gaussian beam propagating in weak Kolmogorov turbulence assumes values in the range $1 \leq \rho_{0\_G} \leq 1.8$. Consequently we deduce that the same range of values stays valid for weak oceanic turbulence (under approximation).

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Also, we deduce from Eq. (25) that the coherence radius of a Gaussian beam propagating in weak non-Kolmogorov turbulence assumes values in the range $1 \leq \rho_{0\_G}(\alpha) \leq (\alpha - 1)^{\frac{1}{\alpha-2}}$.

Finally, we mention here two recent papers showing the equivalent refractive index structure constant in non-Kolmogorov turbulence [8] and how to express oceanic turbulence parameters by atmospheric structure constant [9].

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