# Homogenization for viscoelasticity of the integral type with aging and shrinkage

Julia Orlik

ITWM
Erwin-Schroedinger-Str.
67663 Kaiserslautern
Germany

Telefon: +49 (0)631 205 2742
Fax: +49 (0)631 205 4447
e-mail: orlik@itwm.uni-kl.de

07.07.98

## Introduction

A multi-phase composite with periodic distributed inclusions with a smooth boundary is considered in this contribution. The composite component materials are supposed to be linear viscoelastic and aging (of the non-convolution integral type, for which the Laplace transform with respect to time is not effectively applicable) and are subjected to isotropic shrinkage. The free shrinkage deformation can be considered as a fictitious temperature deformation in the behavior law.
The procedure presented in this paper proposes a way to determine average (effective homogenized) viscoelastic and shrinkage (temperature) composite properties and the homogenized stress-field from known properties of the components. This is done by the extension of the asymptotic homogenization technique known for pure elastic non-homogeneous bodies to the non-homogeneous thermo-viscoelasticity of the integral non-convolution type.
Up to now, the homogenization theory has not covered viscoelasticity of the integral type. Sanchez-Palencia (1980), Francfort & Suquet (1987) (see [2], [9]) have considered homogenization for viscoelasticity of the differential form and only up to the first derivative order. The integral-modeled viscoelasticity is more general then the differential one and includes almost all known differential models.
The homogenization procedure is based on the construction of an asymptotic solution with respect to a period of the composite structure. This reduces the original problem to some auxiliary boundary value problems of elasticity and viscoelasticity on the unit periodic cell, of the same type as the original non-homogeneous problem. The existence and uniqueness results for such problems were obtained for kernels satisfying some constrain conditions. This is done by the extension of the Volterra integral operator theory to the Volterra operators with respect to the time, whose

kernels are space linear operators for any fixed time variables. Some ideas of such approach were proposed in [11] and [12], where the Volterra operators with kernels depending additionally on parameter were considered. This manuscript delivers results of the same nature for the case of the space-operator kernels.

## Notations and definitions

First of all we want to introduce some sets, spaces and classes used in the work.

$\mathbb{R}^n$ is the eucledean space of n-dimensional column vectors.
$\mathbb{R}^{n\times m}$ is the set of $(n \times m)$-dimensional matrices with real entries and a norm associated with the corresponding norms in $\mathbb{R}^m$ and $\mathbb{R}^n$.
$Y$ is a periodicity cell in $\mathbb{R}^n$, $Y := ]0, Y_{01}[\times ... \times ]0, Y_{0n}[, \quad Y_{01}, ..., Y_{0n} \in ]0, \infty[$
$J$ is a real axis segment $[t_1, t_2], \quad 0 \le t_1 < t_2 < \infty$
$\Big\langle F \Big\rangle \quad := \frac{1}{\lambda(Y)} \int_Y F d\lambda$, where $\lambda$ is a measure on $Y$.
$C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with a compact support in $\Omega$
$H^1(\Omega, \gamma)$ is the closure with respect to the $H^1$-norm of the subspace $C^\infty(\bar{\Omega})$ formed by all functions vanishing in a neighbourhood of $\gamma$, where $\gamma \subset \partial\Omega$. This space is associated with the Dirichlet boundary conditions: $H^1(\Omega, \gamma) := \{u \in H^1(\Omega) : \ u|_\gamma = 0\}$.

We will use here some non-standard notations for periodical functions spaces, which were proposed in [7]:

$F_{per}^Y$ is the set of all $Y$-periodic functions.

$C_{per}^k(Y) \quad := \{\phi : \ Y \to \mathbb{R} : \ \exists f \in C^k(\mathbb{R}^n) \bigcap F_{per}^Y | \ \phi = f$ restricted to $Y\}$.

$C_{per}^\infty(Y) \quad := \bigcap_{k\in\mathbb{N}_0} C_{per}^k(Y)$.

$C_B^1(\bar{Y}) \quad := \{\phi : Y \to \mathbb{R}, \ \phi$ is uniformly continuous and differentiable on $\bar{Y}\}$.

$H_{per}^1(Y)$ is the closure in $H^1(Y)$ of $C_{per}^0(Y) \bigcap C_B^1(\bar{Y})$

$L^1_{[0]}(Y) \quad := \{f \in L^1(Y) : < f >= 0\}.$

$H^1_{[0]}(Y) \quad := H^1(Y) \bigcap L^1_{[0]}(Y).$

$H^1_{per[0]}(Y) \quad := H^1_{per}(Y) \bigcap H^1_{[0]}(Y).$

**Definition 1** *Let $X$, $Z$ be Banach spaces. $\mathcal{L}(X, Z)$ is the space of all linear continuous operators from $X$ to $Z$, $K : X \to Z$, with the norm*

$$||K||_{\mathcal{L}(X,Z)} := \sup_{||x||_X \leq 1} ||Kx||_Z.$$

*If $X = Z$, $\mathcal{L}(X) := \mathcal{L}(X, Z)$*

**Definition 2** *$C(J; X)$ is a space of functions $f(t)$ mapping the real axis segment $J$ into a Banach space $X$:*

$$f : \quad J \to X \quad ,$$

*which are $|| \cdot ||_X$-continuous:*

$$||f(t + \Delta t) - f(t)||_X \to 0 \qquad \text{as } \Delta t \to 0 \ \ \forall t \in J$$

*and are bounded with respect to their norm in $C(J; X)$:*

$$||f||_{C(J;X)} := \sup_{t \in J} ||f(t)||_X < \infty.$$

**Definition 3** $V(C; J; \mathbb{R}^{n\times n})$ *is the class of Volterra integral operators*

$$A\star : \quad (A \star g)(t) := \int_J A(t, \tau)g(\tau)d\tau, \ \ t, \tau \in J, \ \ A(t, \tau) = 0 \ \ \forall \tau > t,$$

*such that*

*(i) the kernels $A : \quad J^2 \to \mathbb{R}^{n\times n}$ are matrix-functions $A(t, \tau) = \{A^{hk}(t, \tau)\}, \ h, k = 1, ..., n$ and $g(t)$ is a vector-function.*
*(ii)*

$$\int_0^t |A(t + \Delta t, \tau) - A(t, \tau)|d\tau \to 0 \ \ \text{as} \ \ \Delta t \to 0 \ \ \forall t \in J$$

*(iii) the kernel norm is bounded:*

$$|||A|||_{V(C;J;\mathbb{R}^{n\times n})} := \sup_{t \in J} \int_0^t |A(t, \tau)|d\tau < \infty.$$

Here $|A|$ denotes a matrix norm.

**Example 4** *Often, the kernels $A(t,\tau)$ are of the convolution type and taken in the exponential form:*

$$A(t,\tau) = \sum_{i=1}^{m} \alpha_i e^{-\beta_i(t-\tau)},$$

*where $\alpha_i, \beta_i$ are constants.*
*$A\star \in V(C; J; \mathbb{R}^{n\times n})$ may have also kernels of the Abel type (typical example for the relaxation kernels of concrete and cement [13], [14]):*

$$A(t,\tau) = A_c(t,\tau)(t-\tau)^{-\alpha} + B_c(t,\tau)(\tau)^{-\beta} + A_c^*(t,\tau), \;\; 0 \le \alpha, \beta < 1,$$

*$A_c$, $B_c$ and $A_c^*$ are bounded continuous function with respect to $t$, $\tau$.*

Let us define a class of matrix operators parametrically dependent on the space coordinate $x \in \Omega$.

**Definition 5** $V(C; \Omega; J; \mathbb{R}^{n\times n})$ *is the class of Volterra integral operators*

$$A\star : \;\; (A \star g)(x,t) := \int_J A(x,t,\tau) g(x,\tau) d\tau, \;\; x \in \Omega, \;\; t, \tau \in J,$$

$$|A(x,t,\tau)| = 0 \;\; \forall \tau > t, \forall x \in \Omega,$$

*such that*
*(i) the kernels $A: \;\; \Omega \times J^2 \to \mathbb{R}^{n\times n}$ are matrix-functions $A(x,t,\tau) = (A^{hk}(x,t,\tau))_{n\times n}$, $g(x,t)$ is a vector-function,*
*(ii)*

$$\int_0^t \sup_{x\in\Omega} |A(x,t+\Delta t,\tau) - A(x,t,\tau)| d\tau \to 0 \;\; as \;\; \Delta t \to 0 \;\; \forall t \in J,$$

*(iii) the kernel norm is finite*

$$|||A|||_{V(C;\Omega;J;\mathbb{R}^{n\times n})} := \sup_{t\in J} \int_0^t \sup_{x\in\Omega} |A(x,t,\tau)| d\tau < \infty.$$

*Here $|A|$ denotes a matrix norm.*

**Example 6** *For concrete or cement composites the relaxation operator $A(x)\star \in V(C; \Omega; J; \mathbb{R}^{n\times n})$ may have kernels of the type:*

$$A(x,t,\tau) = A_c(x,t,\tau)(t-\tau)^{-\alpha} + B_c(x,t,\tau)(\tau)^{-\beta} + A_c^*(x,t,\tau), \;\; 0 \le \alpha, \beta < 1,$$

*$A_c$, $B_c$ and $A_c^*$ are bounded continuous functions with respect to $t$, $\tau$ and piecewise continuous with respect to the space variable $x$.*

Now we define a class of Banach-valued operators.

**Definition 7** $V(C;J;\mathcal{L}(X,Z))$ *is the class of Volterra integral operators*

$$K_x\star: \ (K_x\star g)(t) := \int_J K_x(t,\tau)g(\tau)d\tau,\ \ t,\tau\in J,\ \ K_x(t,\tau)=0\ \forall\tau>t,$$

*such that*

*(i) with operator kernels* $K_x:\ \ J^2\to\mathcal{L}(X,Z)$, *and* $K_x(t,\tau)g(\tau)\in B^1(J,Z)$ *(Bochner integrable, see Yosida [16] for definition),*

*(ii)* $K_x\star$ *are* $||\cdot||_{\mathcal{L}(X,Z)}$*-continuous, i.e.,*

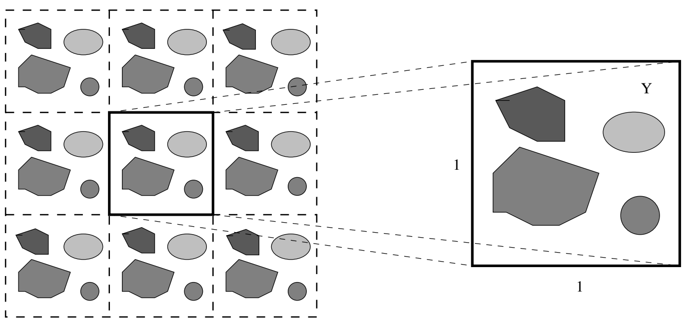
$$\int_0^t ||K_x(t+\Delta t,\tau)-K_x(t,\tau)||_{\mathcal{L}(X,Z)}d\tau\to 0\ \ as\ \ \Delta t\to 0\ \ \forall t\in J,$$

*(iii) and* $K_x\star$ *have finite kernel norms:*

$$|||K_x|||_{V(C;J;\mathcal{L}(X,Z))} := \sup_{t\in J}\int_0^t ||K_x(t,\tau)||_{\mathcal{L}(X,Z)}d\tau<\infty.$$

## Non-homogeneous problem

As said before, the non-homogeneous solid under consideration is composed of n isotropic viscoelastic materials. It occupies a bounded domain $\Omega\subset R^3$ with a Lipschitz boundary $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$ and such that $\partial\Omega_1,\partial\Omega_2$ are mutually disjoint subsets of $\partial\Omega$. We assume that this solid has a periodic structure (figure below) with a period $\varepsilon$ (a scaling parameter). Let us consider the equilibrium equations:



$$\frac{\partial}{\partial x_h}\Big[\Big({a_{ij}^{hk}}^{\varepsilon}_0(x,t)+{a_{ij}^{hk}}^{\varepsilon}(x)\star\Big)\frac{\partial u_j^{\varepsilon}(x,t)}{\partial x_k}\Big]=f_{i0}(x,t)+\frac{\partial {f_i^h}^{\varepsilon}(x,t)}{\partial x_h},\quad x\in\Omega \tag{1}$$

$$i,j,h,k=1,2,3$$

with boundary conditions:

$$u_i^{\varepsilon}(x,t)=0\quad \forall t\in[0;T]\qquad \text{on } \partial\Omega_1, \tag{2}$$

$$\Big[\Big({a_{ij}^{hk}}^{\varepsilon}_0(x,t)+{a_{ij}^{hk}}^{\varepsilon}(x)\star\Big)\frac{\partial u_j^{\varepsilon}(x,t)}{\partial x_k}\Big]n_h={f_i^h}^{\varepsilon}(x,t)n_h \text{ on } \partial\Omega_2, \tag{3}$$

where

$$({a_{ij}^{hk}}^{\varepsilon}(x)\star e_k^j)(t)=\int\limits_0^t {a_{ij}^{hk}}^{\varepsilon}(x,t,\tau)\cdot e_k^j(x,\tau)d\tau, \tag{4}$$

$u_i^{\varepsilon}(\cdot,t)\in H^1(\Omega,\partial\Omega_1)$, Volterra integral operators ${a_{ij}^{hk}}^{\varepsilon}(x)\star\in V(C;\Omega;[0,T];\mathbb{R}^{n\times n})$ and their kernels ${a_{ij}^{hk}}^{\varepsilon}(x,t,\tau)$, like elastic coefficients ${a_{ij}^{hk}}^{\varepsilon}_0(x,t)$ of the free term, belong to $F_{per}^Y\bigcap L^{\infty}(\Omega)$ with respect to $x$ for almost all fixed $t,\tau\in[0,T]$, $f_{i0}$ is a vector of the external forces, $f_{i0}(\cdot,t)\in L^2(\Omega)$, ${f_i^h}^{\varepsilon}$ is a so-called shrinkage stress tensor, ${f_i^h}^{\varepsilon}(\cdot,t))\in F_{per}^Y\bigcap L^2(\Omega)$. All functions are supposed to be in $C([0,T])$ with respect to the time variable $t$ a.e. in $\Omega$.
Let us introduce a notation ${\underline{a}_{ij}^{hk}}^{\varepsilon}(x,t):={a_{ij}^{hk}}^{\varepsilon}_0(x,t)+{a_{ij}^{hk}}^{\varepsilon}(x)\star$. We will refer to a unit cell $Y=]0,1[^3$. ${\underline{a}_{ij}^{hk}}^{\varepsilon}(x,t):=\underline{a}_{ij}^{hk}(\xi,t)$, ${f_i^h}^{\varepsilon}(x,t):=f_i^h(x,\xi,t)$, $(\xi:=\frac{x}{\varepsilon})$. Dependence on $x$ describes outer (macro-) effects, while dependence on $\xi$ describes effects related to the composite structure.
The whole coefficient tensor $(\underline{a}_{ij}^{hk}(\xi,t))$ is assumed to be symmetric at each point $\xi\in Y$:

$$\underline{a}_{ij}^{hk}(\xi,t)=\underline{a}_{ji}^{kh}(\xi,t)=\underline{a}_{hj}^{ik}(\xi,t)=\underline{a}_{ik}^{hj}(\xi,t),\quad \forall t\in[0;T] \tag{5}$$

and tensor $({a_{ij}^{hk}}_0(\xi,t))$ is additionally positive-definite, with elements bounded at each point $\xi\in Y$ ([2],Ch.6):

$$c_0\eta_k^j\eta_k^j\le {a_{ij}^{hk}}_0(\xi,t)\eta_h^i\eta_k^j\le C_0\eta_k^j\eta_k^j, \tag{6}$$

for all $\eta_k^j$ where the constants $0<c_0\le C_0<\infty$ are independent of $\xi$.

$${\underline{a}_{ij}^{hk}}^{\varepsilon}(x)=\begin{cases} {\underline{a}_{ij}^{hk}}^{s} & \text{if } x\in\Omega_s\\ {\underline{a}_{ij}^{hk}}^{M} & \text{if } x\in\Omega\backslash\bigcup_{s=1}^m\Omega_s\end{cases}, \tag{7}$$

where $s=1,...,m$, $m$ is the number of inclusions from the different materials. For isotropic materials

$${\underline{a}_{ij}^{hk}}^{\varepsilon}=\underline{\lambda}^{\varepsilon}\delta_{hi}\delta_{kj}+\underline{\mu}^{\varepsilon}\delta_{ij}\delta_{hk}+\underline{\mu}^{\varepsilon}\delta_{ik}\delta_{hj}. \tag{8}$$

## Existence and uniqueness in inhomogenious viscoelasticity of the integral type

Now we want to prove the existence and uniqueness for the solution to the system of equations:

$$\frac{\partial}{\partial x_h}\Big[\Big(A_0^{hk}(x,t)+A^{hk}(x)\star\Big)\frac{\partial u(x,t)}{\partial x_k}\Big]=f_0(x,t)+\frac{\partial f_h(x,t)}{\partial x_h},\qquad x\in\Omega \quad (9)$$

with boundary conditions:

$$u(x,t)=0 \text{ on } \partial\Omega_1,\quad \forall t\in[0;T] \quad (10)$$

$$\Big[\Big(A_0^{hk}(x,t)+A^{hk}(x)\star\Big)\frac{\partial u(x,t)}{\partial x_k}\Big]n_h=f_h(x,t)n_h \text{ on } \partial\Omega_2, \quad (11)$$

where $\Omega\subset\mathbb{R}^n$. We use the following notation in this section: $A^{hk}(x,t):=(a_{ij}^{hk}(\frac{x}{\varepsilon},t))$ and $f_h(x,t):=(f_i^h(x,\frac{x}{\varepsilon},t))$.
Let us assume $u(t)\in C([0,T];H^1(\Omega,\partial\Omega_1))$, $f_h(t)\in C([0,T];L^2(\Omega))$, $h=0,...,n$.
Further, we introduce the following notation:

$$A_{0x}(t)u:=\frac{\partial}{\partial x_h}\Big(A_0^{hk}(x,t)\frac{\partial u(x,t)}{\partial x_k}\Big), \quad (12)$$

$$A_x(t,\tau)u:=\frac{\partial}{\partial x_h}\Big(A^{hk}(x,t,\tau)\frac{\partial u(x,\tau)}{\partial x_k}\Big). \quad (13)$$

$A_{0x}(t),A_x(t,\tau):H^1(\Omega,\partial\Omega_1)\to H^{-1}(\Omega,\partial\Omega_2)$ for each fixed $t$ and $\tau$. Here by $H^{-1}(\Omega,\partial\Omega_2)$ is denoted the space dual to $H^1(\Omega,\partial\Omega_1)$.
Now, let us rewrite the equation (9) with respect to the notation above:

$$A_{0x}(t)u(t)+\int_0^t A_x(t,\tau)u(\tau)d\tau=f_0(t)+\frac{\partial f_h(t)}{\partial x_h}, \quad (14)$$

or, if we refer to the $\star$-notation:

$$\Big(A_{0x}(t)+A_x\star\Big)u(t)=f_0(t)+\frac{\partial f_h(t)}{\partial x_h}. \quad (15)$$

A variational equality for (9)-(11) has the following form:

$$\begin{aligned}&\int_\Omega\Big(A_{0x}+A_x\star\Big)uv\,dx\\ &=-\int_\Omega\Big(A_0^{hk}+A^{hk}\star\Big)\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_h}dx+\int_{\partial\Omega_2}\Big(A_0^{hk}+A^{hk}\star\Big)\frac{\partial u}{\partial x_k}vn_h ds \quad (16)\\ &\qquad=\int_\Omega\Big(f_0v-f_h\frac{\partial v}{\partial x_h}\Big)dx+\int_{\partial\Omega_2}f_hvn_hds,\end{aligned}$$

$$\forall v \in C([0,T]; H^1(\Omega, \partial\Omega_1)).$$

If we take in the consideration condition (11) the variational identity can be rewritten as

$$\int_\Omega \Big(A_0^{hk} + A^{hk}\star\Big)\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_h}dx = \int_\Omega \Big(-f_0 v + f_h \frac{\partial v}{\partial x_h}\Big)dx, \tag{17}$$

$$\forall v \in C([0,T]; H^1(\Omega, \partial\Omega_1)).$$

**Theorem 8** *Let $\Omega$ be a domain with an inhomogeneous structure and let the coefficient matrix of the operator $A_{0x}(t)$ satisfies the positivity and continuity condition (6) with constants $c_0, C_0 > 0$. Furthermore, let $f_h \in C([0,T]; L^2(\Omega))$, $h = 0, ..., n$. Then, if the operator $A_x\star$ belongs to $V(C; [0,T]; \mathcal{L}(H^1(\Omega, \partial\Omega_1), H^{-1}(\Omega, \partial\Omega_2)))$, there exists a unique global weak solution of the problem (9)-(11) in $C([0,T]; H^1(\Omega, \partial\Omega_1))$.*

Proof
Due to the Theorem 2.7 [3] (Chap.1) Korn's inequality

$$||v||_{H^1(\Omega)} \leq C||e(v)||_{L^2(\Omega)}, \tag{18}$$

where $e(v) := (e_k^j(v))_{n\times n}$, $e_k^j(v) = \frac{1}{2}\Big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\Big)$ and

$$||e(v)||^2_{L^2(\Omega)} = \frac{1}{2}||\nabla v||^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_j}dx,$$

holds for all $v \in H^1(\Omega, \partial\Omega_1)$. Since conditions (18) and (6) are satisfied, the bilinear form $a_{ij\,0}^{hk} e_h^i e_k^j \equiv a_{ij\,0}^{hk}\frac{\partial v_i}{\partial x_h}\frac{\partial v_j}{\partial x_k} \equiv A_0^{hk}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_h}$ in (17) satisfies all assumptions of the Lax-Milgram theorem for $H^1(\Omega, \partial\Omega_1)$, and hence $A_{0x}$ is invertible with

$$||A_{0x}(t)|| \leq C_0 \qquad \text{and} \qquad ||A_{0x}^{-1}(t)|| \leq \frac{1}{c_0} \tag{19}$$

uniformly with respect to $t$.
We can rewrite the equation (1) with respect to the notations introduced in the following way:

$$u(t) + A_{0x}^{-1}(t)\int_0^t A_x(t,\tau)u(\tau)d\tau = A_{0x}^{-1}(t)(f_0(t) + \frac{\partial f_h(t)}{\partial x_h}). \tag{20}$$

Observe that $A_{0x}^{-1}(t)$ does not depend on $\tau$.

$$u(t) + \int_0^t A_{0x}^{-1}(t)A_x(t,\tau)u(\tau)d\tau = A_{0x}^{-1}(t)(f_0(t) + \frac{\partial f_h(t)}{\partial x_h}). \tag{21}$$

The following new notation is

$$K_x(t,\tau) := {A_0}_x^{-1}(t)A_x(t,\tau), \tag{22}$$

where $K_x(t,\tau) : H^1(\Omega,\partial\Omega_1) \to H^1(\Omega,\partial\Omega_1)$ for each fixed $t$ and $\tau$, or $K_x(x,t,\tau) : H^1(\Omega,\partial\Omega_1) \times [0;T]^2 \to H^1(\Omega,\partial\Omega_1)$. We are looking for a resolvent (and its existence) for the following equation:

$$u(t) + \int_0^t K_x(t,\tau)u(\tau)d\tau = {A_0}_x^{-1}(t)(f_0(t) + \frac{\partial f_h(t)}{\partial x_h}), \tag{23}$$

or by using the $\star$ -notation

$$u(t) + K_x \star u(t,\tau) = F(t). \tag{24}$$

Here

$$F(t) := {A_0}_x^{-1}(t)(f_0(t) + \frac{\partial f_h(t)}{\partial x_h}), \tag{25}$$

$F(t) : [0,T] \to H^1(\Omega,\partial\Omega_1)$.
Let us suppose, there exists a solution to (24) and that it is represented formal by a Neumann series

$$u(t) = \Big(1 + K_x \star\Big)^{-1} F(t) = \sum_{j=0}^{\infty}(-1)^j \Big[(K_x\star)^j F\Big](t). \tag{26}$$

Now, our aim is to show, that this series converges. We are working in the Banach space. Banach spaces are complete, it means that each Cauchy sequence converges. Hence, it is enough to show that

$$S_n := \sum_{j=0}^{n}(-1)^j \Big[(K_x\star)^j F\Big](t) \tag{27}$$

is a Cauchy sequence, i.e. that for any $\varepsilon > 0$ there exists a number $N$ such that $||S_n - S_m||_{H^1(\Omega,\partial\Omega_1)} < \varepsilon$ for all $m, n \geq N$.
We can estimate

$$||S_n - S_m||_{H^1(\Omega,\partial\Omega_1)} = ||\sum_{j=m}^{n}(-1)^j \Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$$

$$\leq \sum_{j=m}^{\infty} ||(-1)^j \Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} + \sum_{j=n}^{\infty} ||(-1)^j \Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$$

$$\leq 2\sum_{j=N}^{\infty} ||(-1)^j \Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \quad \text{for } N \leq m,n.$$

If we will be able to find such a $N$ that $\sum_{j=N}^{\infty}||(-1)^j\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} < \varepsilon/2$ we are in business. The last statement is valid if the function series $\sum_{j=0}^{\infty}||\Big[(K_x\star)^j\ F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$ converges.
We formulate the following lemma now, which we will need later.

**Lemma 9** *Let us denote* $\phi(t)=\int_0^t K_x(t,\tau)f(\tau)d\tau$. *Then*

$$||\phi(t)||^2_{H^1(\Omega)} \leq 2\{\int_0^t ||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega))}||f(\tau)||_{H^1(\Omega)}d\tau\}^2$$
$$\leq 2\{|||K_x|||_{V(C;[0,T];\mathcal{L}(H^1))}\sup_{\tau\in J}||f(\tau)||_{H^1(\Omega)}\}^2,$$

*where*

$$|||K_x|||_{V(C;[0,T];\mathcal{L}(H^1))} := \sup_{t\in J}\int_0^t ||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega))}$$

*and*

$$||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega))} := \sup_{||f(\tau)||_{H^1(\Omega)}\leq 1} ||K_x(t,\tau)f(\tau)||_{H^1(\Omega)}.$$

Proof of Lemma:
On the basis of Minkowski inequality

$$||\phi(t)||^2_{H^1(\Omega)} := \int_\Omega |\phi(t)|^2dx + \int_\Omega |D_x\phi(t)|^2dx$$
$$= \int_\Omega |\int_0^t K_x(t,\tau)f(\tau)d\tau|^2dx + \int_\Omega |\int_0^t D_xK_x(t,\tau)f(\tau)d\tau|^2dx$$
$$\leq \{\int_0^t[\int_\Omega |K_x(t,\tau)f(\tau)|^2dx]^{1/2}d\tau\}^2 + \{\int_0^t[\int_\Omega |D_xK_x(t,\tau)f(\tau)|^2dx]^{1/2}d\tau\}^2$$
$$\leq \{\int_0^t([\int_\Omega |K_x(t,\tau)f(\tau)|^2dx]^{1/2} + [\int_\Omega |D_xK_x(t,\tau)f(\tau)|^2dx]^{1/2})d\tau\}^2$$
$$\leq 2\{\int_0^t([\int_\Omega |K_x(t,\tau)f(\tau)|^2dx + \int_\Omega |D_xK_x(t,\tau)f(\tau)|^2dx]^{1/2})d\tau\}^2$$
$$\leq 2\{\int_0^t ||K_x(t,\tau)f(\tau)||_{H^1(\Omega)}d\tau\}^2 \leq 2\{\int_0^t ||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega))}||f(\tau)||_{H^1(\Omega)}d\tau\}^2$$
$$\leq 2\{|||K_x|||_{V(C;[0,T];\mathcal{L}(H^1))}(\sup_{\tau\in J}||f(\tau)||_{H^1(\Omega)}\}^2 \quad .$$

Lemma is proven.
We should remark, that $||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}$ can be estimated by the norm product of the operators $A_{0x}^{-1}(t)$ and $A_x(t,\tau)$ in corresponding spaces:

$$||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))} \leq \frac{1}{c_0}||A_x(t,\tau)||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1),H^{-1}(\Omega,\partial\Omega_2))} \tag{28}$$

Let us show the convergence of the series $\sum_{j=0}^{\infty}||\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$ majorizing the Banach-valued Neumann's series for $u(t)$ in (26):

$$\begin{aligned}
||u(t)||_{H^1(\Omega,\partial\Omega_1)} &= ||\sum_{j=0}^{\infty}(-1)^j\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \\
&\leq \sum_{j=0}^{\infty}||\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \\
&= \sum_{j=0}^{\infty}||\Big[K_x\star\Big[K_x\star\Big[...\Big[K_x\star F\Big]\Big]\Big]\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \\
&= \sum_{j=0}^{\infty}||\int_0^t K_x(t,t_{j-1})\int_0^{t_{j-1}} K_x(t_{j-1},t_{j-2})... \\
&\quad ...\int_0^{t_1} K_x(t_1,\tau)F(\tau)d\tau...dt_{j-2}dt_{j-1}||_{H^1(\Omega,\partial\Omega_1)} \\
&= \sum_{j=0}^{\infty}||\Big[K_x\star\Big[(K_x\star)^{j-1}F\Big]\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \\
&\overset{Lemma}{\leq} \sum_{j=0}^{\infty}\sqrt{2}||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star||\Big[(K_x\star)^{j-1}F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \\
&\leq \sum_{j=0}^{\infty}(\sqrt{2}||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star)^j||F(t)||_{H^1(\Omega,\partial\Omega_1)} \qquad (29)
\end{aligned}$$

Since $f_h \in C([0,T];L^2(\Omega))$, $h=0,...,n$, the second factor in (29) is bounded (see, e.g. [3] (Chap.1, Th.5.1)):

$$\begin{aligned}
||F(\tau)||_{H^1(\Omega,\partial\Omega_1)} = ||{A_0}_x^{-1}((f_0(\tau)+\frac{\partial f_h(\tau)}{\partial x_h}))||_{H^1(\Omega,\partial\Omega_1)} \qquad (30) \\
\leq C\sum_{h=0}^{n}||f_h(\tau)||_{L^2(\Omega)},
\end{aligned}$$

where C is a positive constant.

We have obtained the result, that the series $\sum_{j=0}^{\infty}||\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$ to be shown convergent is majorized by the Neumann's series for an operator $\sqrt{2}||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star \in V(C;[0,T];\mathbb{R}^{1\times 1})$ with the scalar kernel $||K_x(t,\tau)||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}$ :

$[0,T]^2 \to \mathbb{R}$:

$$\sum_{j=0}^{\infty} ||\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)} \leq \sum_{j=0}^{\infty} (\sqrt{2}||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star)^j ||F(t)||_{H^1(\Omega,\partial\Omega_1)}. \tag{31}$$

According to the theorem 9.5.5 (ii) [6], if $||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star \in V(C;[0,T];\mathbb{R}^{k\times k})$ and $[0,T]$ is bounded, i.e. $T<\infty$, the majorizing Neumann's series in the righthandside of (31) converges. Hence the function series $\sum_{j=0}^{\infty} ||\Big[(K_x\star)^j F\Big](t)||_{H^1(\Omega,\partial\Omega_1)}$ converges and hence the Banach-valued Neumann's series for $u(t)$ in (26) also converges.
It remains only to remark that $||K_x||_{\mathcal{L}(H^1(\Omega,\partial\Omega_1))}\star \in V(C;[0,T];\mathbb{R}^{1\times 1})$ since $K_x\star \in V(C;[0,T];\mathcal{L}(H^1(\Omega,\partial\Omega_1)))$ and that is true as soon as $A_x(t) \in V(C;[0,T];$ $\mathcal{L}(H^1(\Omega,\partial\Omega_1),\ H^{-1}(\Omega,\partial\Omega_2)))$ and condition (19) is satisfied.

**Remark 10** *We just remind that actually, the solution $u$ interpreted as $u(t)$ : $[0,T]\to H^1(\Omega,\partial\Omega_1)$ can be a real valued vector-function of two variables $x\in\Omega$ and $t\in[0,T]$, i.e. $u(x,t) : \Omega\times[0,T]\to\mathbb{R}^n$, with $u(\cdot,t)\in H^1(\Omega,\partial\Omega_1)$ $\forall t\in[0,T]$ and $u(x,\cdot)\in C([0,T])$ a.e. in $\Omega$.*

## 2-scale Homogenization. Formal asymptotic expansion.

Now, we study the asymptotic behavior of $u^\varepsilon(x)$ with respect to any small fixed $\varepsilon$. We are looking for asymptotics of the solution of (1) of the form

$$u^\varepsilon(x,t) = \sum_{p=0}^{\infty} \varepsilon^p u_p(x,\xi,t), \qquad \xi\in Y,\ x\in\Omega \tag{32}$$

where $u_p(x,\xi,t)$ are vector-functions 1-periodic with respect to $\xi=\frac{x}{\varepsilon}$. After the substitution of (32) into the equilibrium equations and after using these equations for the terms of orders $\varepsilon^{-2},\varepsilon^{-1},\varepsilon^{0}$ separately, we get the following conditions

$$\varepsilon^{-2}: \qquad \frac{\partial}{\partial\xi_h}\Big(A_0^{hk}(\xi,t)\frac{\partial u_0(x,\xi,t)}{\partial\xi_k}\Big) = 0, \quad u_0(\cdot,t)\in H^1_{per[0]}(Y) \tag{33}$$

$$\varepsilon^{-1}: \qquad \frac{\partial}{\partial\xi_h}\Big[\Big(A_0^{hk}(\xi,t)+A^{hk}(\xi)\star\Big)\Big(\frac{\partial u_0(x,t)}{\partial x_k}+\frac{\partial u_1(x,\xi,t)}{\partial\xi_k}\Big)\Big] \tag{34}$$

$$= \frac{\partial f_h(x,\xi,t)}{\partial\xi_h}, \quad u_1(\cdot,t)\in H^1_{per}(Y)$$

$$\varepsilon^{0}: \qquad \frac{\partial}{\partial x_h}\Big\langle\Big(A_0^{hk}(\xi,t)+A^{hk}(\xi)\star\Big)\Big(\frac{\partial u_0(x,t)}{\partial x_k}+\frac{\partial u_1(x,\xi,t)}{\partial\xi_k}\Big)\Big\rangle \tag{35}$$

$$= f_0(x,t) + \frac{\partial < f_h(x,\xi,t) >}{\partial x_h}, \quad u_0(\cdot,t) \in H^1(\Omega, \partial\Omega_1)$$

In the standard homogenization procedure, the equation (34) is being used for the first asymptotic approximation $u_1(x,\xi,t)$ expression in the term of the homogeneous function $u_0(x,t)$. The following substitution is known ([1], [2]) for pure elastical case :

$$u_1(x,\xi) := -N_{0p}(\xi)\frac{\partial u_0(x)}{\partial x_p} + y(x,\xi), \tag{36}$$

where $N_{0p}(\xi), y(x,\xi)$ are auxiliary unknown 1-periodic with respect to $\xi$ tensor-functions.
Let us try to find a similar substitution for our case. First, consider (34) with a homogeneous right-handside. The weak formulation for such a problem has a form

$$\int_Y \Big[\Big(A_0^{hk}(\xi,t) + A^{hk}(\xi)\star\Big)\Big(\frac{\partial u_1(x,\xi,t)}{\partial \xi_k} + \frac{\partial u_0(x,t)}{\partial x_k}\Big)\Big]\frac{\partial v(\xi,t)}{\partial \xi_h}d\xi = 0, \tag{37}$$

$$\forall v \in H^1_{per}(Y).$$

We introduce the following scalar product in the space $H^1_{per[0]}$:

$$(u(t),v(t))_{H^1_{per[0]}} = \int_Y A_0^{hk}(\xi,t)\frac{\partial u(\xi,t)}{\partial \xi_k}\frac{\partial v(\xi,t)}{\partial \xi_h}d\xi, \quad \forall v(\cdot,t) \in H^1_{per[0]} \tag{38}$$

which is associated to the original norm in $H^1$.
Moreover, since $A_0^{hk}$ satisfy the condition (6) and if $B$ is a linear continuous operator from $H^1_{per[0]}$ to $H^1_{per[0]}$ for each fixed pair $t$, $\tau$ and $N_{0h}(\xi,t) \in C([0,T]; H^1_{per[0]})$, then we define them by

$$(B(t,\tau)u(\tau), v(\tau))_{H^1_{per[0]}}$$

$$= \int_Y A^{hk}(\xi,t,\tau)\frac{\partial u(\xi,\tau)}{\partial \xi_k}\frac{\partial v(\xi,\tau)}{\partial \xi_h}d\xi, \quad \forall v(\cdot,\tau) \in H^1_{per[0]} \tag{39}$$

$$(N_{0h}(t), v(t))_{H^1_{per[0]}} = \int_Y A_0^{hk}(\xi,t)\frac{\partial v(\xi,t)}{\partial \xi_k}d\xi, \quad \forall v(\cdot,t) \in H^1_{per[0]} \tag{40}$$

Now we can rewrite equation (37) in the terms of the new notation:

$$(u_1(t),v(t))_{H^1_{per[0]}} + \int_0^t (B(t,\tau)u_1(\tau),v(\tau))_{H^1_{per[0]}}d\tau + (N_{0h}(t)\frac{\partial u_0(x,t)}{\partial x_h}, v(t))_{H^1_{per[0]}} \tag{41}$$

$$+\int_0^t (B(t,\tau)N_{0h}(\tau)\frac{\partial u_0(x,\tau)}{\partial x_h}, v(\tau))_{H^1_{per[0]}} d\tau = 0 \quad \forall v(\cdot,t) \in H^1_{per[0]}.$$

If we put out $v(t)$ in this equation, and keep in mind commutation of the integration concerning time with a space scalar product, linearity of this product and arbitrarity of $v$, we obtain that the following condition has to be equal to zero:

$$u_1 + B \star u_1 + N_{0h}\frac{\partial u_0(x)}{\partial x_h} + B \star \frac{\partial u_0(x)}{\partial x_h} = 0 \tag{42}$$

It is known from the resolvent theory, if the operator $B\star$ is bounded in some sense (this was already discussed in the previous section), then there exists a resolvent $R\star$ to the integral operator $B\star$ such that the solution of (42) can be found as

$$u_1 = -N_{0h}\frac{\partial u_0(x)}{\partial x_h} - B \star \frac{\partial u_0(x)}{\partial x_h} + R \star \Big(N_{0h}\frac{\partial u_0(x)}{\partial x_h} + B \star \frac{\partial u_0(x)}{\partial x_h}\Big) \tag{43}$$

By taking into account, that the class of kernels in the consideration forms a Banach algebra with product $\star$ (according to the Theorem5 from the previous section) and applying the multiplication rule:

$$u_1 = -N_{0h}\frac{\partial u_0(x)}{\partial x_h} + \Big( - B + RN_{0h} + R \star B\Big) \star \frac{\partial u_0(x)}{\partial x_h}, \tag{44}$$

we can extract the following structure for the substitution we are looking for:

$$u_1(x,\xi,t) := (-N_{0h}(\xi,t) + N_h(\xi)\star)\frac{\partial u_0(x,t)}{\partial x_h}, \tag{45}$$

where $N_{0h}(\xi,t)$ and $N_h(\xi,t,\tau)$ are auxiliary functions, which are 1-periodic in space coordinate. $N_{0h}(\xi,t)$ is the solution of the equation (40). Now remember the non-homogeneous right-handside of (34). The total solution of (34) can be found as

$$u_1(x,\xi,t) := (-N_{0p}(\xi,t) + N_p(\xi)\star)\frac{\partial u_0(x,t)}{\partial x_p} + y(x,\xi,t), \tag{46}$$

where auxiliary function $y(x,\xi,t)$, which is 1-periodic in space coordinate, is responsible for the righthandside of (34). By substituting (46) into the equation (34), we get the following cell-problems for determining the auxiliary periodic functions:

$$\frac{\partial}{\partial \xi_h}\Big[A_0^{hk}(\xi,t)\frac{\partial\Big(N_{0p}(\xi,t) - \xi_p E\Big)}{\partial \xi_k}\Big] = 0, \qquad \xi \in Y, \tag{47}$$

$$\frac{\partial}{\partial \xi_h}\Big[\Big(A_0^{hk}(\xi,t) + A^{hk}(\xi)\star\Big)\frac{\partial N_p(\xi)}{\partial \xi_k}(t,\tau)\Big] \tag{48}$$
$$= \frac{\partial}{\partial \xi_h}\Big[A^{hk}(\xi,t,\tau)\frac{\partial\Big(N_{0p}(\xi,t) - \xi_p E\Big)}{\partial \xi_k}\Big], \quad \xi \in Y,$$

$$\frac{\partial}{\partial \xi_h}\Big[\Big(A_0^{hk}(\xi,t) + A^{hk}(\xi)\star\Big)\frac{\partial y(\xi,t)}{\partial \xi_k}\Big] = \frac{\partial f_h(\xi,t)}{\partial \xi_h}, \quad \xi \in Y \tag{49}$$

+ periodicity conditions on the cell boundary

+ continuity conditions for displacements and total tractions on the interface inclusion-matrix.

**Remark 11** *The existence and uniqueness of the solution to the equations (48), (49) is given by the Theorem 8 by replacing in its statement $H^1(\Omega, \partial\Omega_1)$ by $H^1_{per[0]}(Y)$. The proof remains. The existence and uniqueness of the solution to (47) in $H^1_{per[0]}(Y)$ for $\forall$ fixed $t \in [0;T]$ can be proven by Lax-Milgram theorem (see Proof of the Theorem 8, (19) or rigorous [1] (Supplement, The.1), or [3] (Chap.1, The. 6.1):*

**Theorem 12** *Consider the following equations system*

$$\frac{\partial}{\partial \xi_h}\Big[A_0^{hk}(\xi)\frac{\partial w(\xi)}{\partial \xi_k}\Big] = F_0(\xi) + \frac{\partial F_h(\xi)}{\partial \xi_h}, \quad \xi \in Y, \tag{50}$$
$$w \in F^Y_{per}, \qquad < w >= 0,$$

*where the vector valued functions $F_h(\xi) \in L^2(Y) \bigcap F^Y_{per}$, $h = 1, ..., n$, the family of matrices $A_0^{hk}(\xi)$ satisfies the condition (6) and their elements $a_{ij}^{hk}(\xi) \in F^Y_{per}$. Let $\Big\langle F_0 \Big\rangle = 0$. Then the problem (50) has a unique solution $w(\xi) \in H^1_{per[0]}(Y)$, and this solution satisfies the estimate*

$$||w||_{H^1(Y)} \leq C(\sum_{h=0}^{n} ||F_h||_{L^2(Y)}), \tag{51}$$

*where the constant $C$ depends only on the constants $c_0$, $C_0$ from the condition (6) and the cell geometry.*

## Homogenized boundary value problem

The average equation system for $u_0(x,t)$ is the following:

$$\frac{\partial}{\partial x_h}\Big[\Big(\hat{A}_0^{hk}(t)+\hat{A}^{hk}\star\Big)\frac{\partial u_0}{\partial x_k}(x,t)-\hat{f}_h(x,t)\Big]=f_0(x,t),\quad x\in\Omega \tag{52}$$

$$u_0(x,t)=0\quad \forall t\in[0;T] \text{ on } \partial\Omega_1 \tag{53}$$

$$\hat{\sigma}_h(x,t)n_h:=\Big[\Big(\hat{A}_0^{hk}(t)+\hat{A}^{hk}\star\Big)\frac{\partial u_0}{\partial x_k}(x,t)-\hat{f}_h(x,t)\Big]n_h=0 \text{ on } \partial\Omega_2, \tag{54}$$

where the averaged coefficients and averaged shrinkage stresses are given by the following expressions:

$$\hat{A}_0^{hp}(t):=\Big\langle {A^{hk}}_0(\xi,t)\frac{\partial\Big(N_{0p}(\xi,t)-\xi_p E\Big)}{\partial\xi_k}\Big\rangle, \tag{55}$$

$$\hat{A}^{hp}\star:=-\Big\langle\Big({A^{hk}}_0(\xi,t)+A^{hk}(\xi)\star\Big)\frac{\partial N_p(\xi)}{\partial\xi_k} \tag{56}$$

$$-\frac{\partial\Big(N_{0p}(\xi,t)-\xi_p E\Big)}{\partial\xi_k}A^{hk}(\xi)\Big\rangle\star$$

$$\hat{f}_h(x,t):=\Big\langle f_h(x,\xi,t)-\Big({A^{hk}}_0(\xi,t)+A^{hk}(\xi)\star\Big)\frac{\partial y(x,\xi,t)}{\partial\xi_k}\Big\rangle. \tag{57}$$

We use here the notation

$$\Big\langle F(\xi)\Big\rangle:=\frac{1}{|Y|}\int_Y F(\xi)d\xi.$$

Let us show, that $\hat{A}_0^{hk}(t)$ satisfies condition (6), $\hat{A}_x^{hk}(t,\tau)\star\in V(C;[0,T];\mathcal{L}(H^1(\Omega,\partial\Omega_1),H^{-1}(\Omega,\partial\Omega_2)))$, where $\hat{A}_x^{hk}(t,\tau)$ is defined in the same way as in (13), and $\hat{f}_h\in C([0,T];L^2(\Omega))$.
Tensor $\hat{A^{hk}}_0(t)$ is calculated exactly in the same way as the one for the pure elastic case and also possesses the conditions of positivity and continuity (6). This proof is given for the pure elastic problem, e.g. in [1] (Chap.4), [2] (Chap. 6). And $\hat{A}_0^{hk}(t)$ belongs obviously to the $C([0,T])$.

Since $N_{0p}(\xi,t)$, $N_p(\xi,t)$ and $y(x,\xi,t)$ are uniformly bounded in $H^1_{per[0]}(Y)$ $\forall t\in[0;T]$ (see Remark.11 or [3], [4]), $\frac{\partial N_{0p}(\xi,t)}{\partial\xi_k}$, $\frac{\partial N_p(\xi,t)}{\partial\xi_k}$ and $\frac{\partial y(\xi,t)}{\partial\xi_k}$ are uniformly bounded for all $t\in[0;T]$ in $L^2(Y)$.
It is easy to show, that the Lemma 9 can be replayed for $L^2$-norm:

**Lemma 13** *Let us denote* $\phi(x,t) = \int_0^t K(t,\tau) f(x,\tau) d\tau$. *Then*

$$
\begin{aligned}
||\phi(.,t)||_{L^2(\Omega)} &\leq \int_0^t |K(t,\tau)| ||f(.,\tau)||_{L^2(\Omega)} d\tau \\
&\leq |||K|||_V \sup_{\tau \in J} ||f(.,\tau)||_{L^2(\Omega)}.
\end{aligned}
$$

Proof of Lemma:
On the basis of Minkowski inequality

$$
\begin{aligned}
||\phi(.,t)||_{L^2(\Omega)} := \Big( \int_\Omega | \int_0^t K(t,\tau) f(x,\tau) d\tau|^2 dx \Big)^{1/2} \\
\leq \int_0^t [\int_\Omega |K(t,\tau) f(x,\tau)|^2 dx]^{1/2} d\tau \\
= \int_0^t |K(t,\tau)| ||f(.,\tau)||_{L^2(\Omega)} d\tau \\
\leq |||K|||_V (\sup_{\tau \in J} ||f(.,\tau)||_{L^2(\Omega)})
\end{aligned}
$$

By applying Lemma 13 to the expressions (56),(**??**) we obviously obtain, that $\hat{A}_x^{hp}(t,\tau)$ is bounded in the norm in $V(C; [0;T]; \mathcal{L}(H^1(\Omega, \partial\Omega_1), H^{-1}(\Omega, \partial\Omega_2)))$ and $\hat{f}_h(t) \in C([0,T]; L^2(\Omega))$.
Let us show now, that (52)-(54) possess a unique solution in $C([0,T]; H^1(\Omega, \partial\Omega_1))$.
For this purpose we only have to reformulate the Theorem 8 for such a problem:

**Theorem 14** *Let $\Omega$ be a domain with a Lipschitz boundary and let the coefficient matrix $\hat{A}_0^{hk}(t)$ satisfies the positivity and continuity condition (6). Furthermore, let $\hat{f}_h(t) \in C([0,T]; L^2(\Omega))$, $h = 0, ..., n$. Then, if the operator $\hat{A}_x \star$ belongs to $V(C; [0,T]; \mathcal{L}(H^1(\Omega, \partial\Omega_1), H^{-1}(\Omega, \partial\Omega_2)))$, there exists a unique global weak solution of the problem (52)-(54) in $C([0,T]; H^1(\Omega, \partial\Omega_1))$.*

The proof of the theorem can be given in a completely analogous way as the one of Theorem 8.

## Conclusion

- Homogenization algorithms and estimates known for the pure elastic non-homogeneous body were obtained for the thermo-elastic problem and for the elastic problem with the additional shrinkage deformation.

- Up to now, the homogenization theory has not covered viscoelasticity of the integral type, but has dealt with viscoelasticity of a rather particular differential form (see [2], [9], [10]).
  Just remind that all viscoelastic differential models of the form

  $$\sum_{i=0}^{r} D_i \overset{(i)}{\sigma} (t) = \sum_{i=0}^{r} B_i \overset{(i)}{e} (t),$$

  where $D_i$, $B_i$ are constants and $\overset{(i)}{\cdot}$ denotes $i$-th times-derivative (see [13], [14]), can be rewritten in the equivalent integral form

  $$\sigma(t) = A(0)e(t) + \int_0^t A(t-\tau)e(\tau)d\tau$$

  with exponential kernels $A(s)$, as shown in the Example 4 (first expression).
  We have proposed the substitution and delivered the homogenization technique for the integral viscoelasticity of the non-convolution type with additional temperature term.

- The existence and uniqueness of the solution of Volterra integral equations concerning the time-variable, with space-operator kernels in Banach spaces was proven. This was done by extension of the classical Volterra integral operator theory with applying the Neumann's series.


## Acknowledgement

This research has been carried out in the framework of the BRITE-EURAM Programme, in the CRAFT/BRITE-EURAM Project BES2-2653, Contract no. BRST-CT97-5198. The financial contribution of the European Commission is highly appreciated.


# References


[1] N.Bakhvalov and G.Panasenko, *Homogenization: Averaging Processes in Periodic Media . Mathematical Problems in the Mechanics of Composite Materials*, Kluwer Academic Publishers Dordrecht/Boston/London, 1984.

[2] E. Sanchez-Palencia *Non Homogeneous Media and Vibration Theory*, Springer-Verlag Berlin/Heidelberg/New York, 1980.

[3] O.A.Oleinik, A.S.Schamaev, G.A.Yosifian *Mathematical problems in elasticity and homogenization*, North-Holland Publishing Company - Amsterdam/London/New York/Tokyo, 1992.

[4] U.Hornung *Homogenization and Porous Media*, Springer, 1997.

[5] W.Hackbusch *Integralgleichungen Theorie und Numerik*, Mathematik, 1997.

[6] G.Gripenberg, S.-O.Londen & O.Staffan *Volterra Integral and Functional Equations* , Cambridge University Press, 1990.

[7] A.Unterreiter *Homogenization of Elliptic PDE*, Lecture material, 1998.

[8] S.Brahim-Otsmane, G.A.Francfort, Francois Murat *Homogenization in thermoelasticity* , in Random Media and Composites, Edited by R.V.Kohn & W.Milton, SIAM, 1988.

[9] G.A.Francfort & P.M.Suquet *Homogenization and Mechanical Dissipation in Thermo-viscoelasticity* , Archive for Rational Mechanics and Analysis, vol.96, no.3, p.265-93, 1986.

[10] K.T.Andrews, K.L.Kuttler, M.Shillor *On the dynamic behaviour of a thermoviscoelastic body in frictional contact with a rigid obstacle*, Euro. Jnl of Applied Mathematics, 1997, vol.8, p.417-436.

[11] S.E.Mikhailov *On some weighted Hardy type classes of one parametric holomorphic functions: II. Partial Volterra operators in parameter*, J. of Mathematical Analysis and Applications, vol.222, p.374-396, 1998

[12] N.Kh.Arutyunyan, B.A.Shoikhet *Asymptotic solution behaviour of the boundary value problem of the theory of creep in inhomogeneously-aging bodies with unilateral relations*, Mech. Solids (Izv. AN SSSR. MTT), vol.16, p.31-48, 1981

[13] Yu.N.Rabotnov, Iljushin *Metoden der Viskoelastizitätstheorie* , Carl Hanser Verlag München, 1970.

[14] Yu.N.Rabotnov *Creep problems in structural members* , Nord-Holland Publishing Company, 1969, Chap.3.

[15] W.Rudin *Real and Complex Analysis*, McGraw-Hill Book Company, 1987, Chap.1.

[16] K.Yosida *Functional Analysis*, Springer-Verlag, 1980.