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Vorwort

Das Tätigkeitsfeld des Fraunhofer Instituts für Techno- und Wirtschaftsmathematik ITWM umfasst anwendungsnahe Grundlagenforschung, angewandte Forschung sowie Beratung und kundenspezifische Lösungen auf allen Gebieten, die für Techno- und Wirtschaftsmathematik bedeutsam sind.

In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe sollen sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen werden.

Darüberhinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation darüber, wie aktuelle Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte transferiert werden, und wie umgekehrt Probleme der Praxis neue interessante mathematische Fragestellungen generieren.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

Optimal Portfolios with Fixed Consumption or Income Streams

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Abstract: We consider some portfolio optimisation problems where either the investor has a desire for an a priori specified consumption stream or/and follows a deterministic pay in scheme while also trying to maximize expected utility from final wealth. We derive explicit closed form solutions for continuous and discrete monetary streams. The mathematical method used is classical stochastic control theory.

Keywords and phrases: Portfolio optimisation, stochastic control, HJB equation, discretisation of control problems.

1 Introduction

The portfolio and consumption problem is a well-studied problem in mathematical finance (see e.g. Karatzas and Shreve (1998), Korn (1997), Merton (1969, 1971, 1990)) and typically consists of maximising the expected utility from terminal wealth and/or consumption until the time horizon of an investor who is endowed with a fixed initial capital. As a non-standard feature we here include the presence of either a fixed pay in scheme or an a priori given deterministic consumption plan of the investor. The first feature can be interpreted as an investor having an investment plan into a fund and the resulting problem then is to determine his best suited investment fund. The second feature simply means that the investor has to finance (parts of) his living expenses from his stock holdings. This problem and also our findings are similar to the results of El Karoui and Jeanblanc-Picqué (1998), but differ in both the methods used and in some aspects of the model. In particular, we will rely on the stochastic control approach via solving the HJB equation explicitly.
We will recall some basic definitions and formulate the problem in section 2. Section 3 will contain explicit results in the case of constant continuous consumption streams. Discrete consumptions will be dealt with in section 4. Section 5 will present a generalization including both cases for income and section 6 will illustrate the behaviour of the portfolio processes.

2 The Model and Some Basic Definitions

We consider a securities market where we have continuous and discrete monetary streams. The continuous streams are modelled by a deterministic, right-continuous and bounded function $c(t)$. The discrete monetary streams takes place at fixed time instants $t_1, \ldots, t_n$ with $0 < t_1 < \ldots < t_n \leq T$ and are given by amounts D_i at times t_i. The continuous streams are introduced for modelling continuous consumption or income streams which are invested at the stock market. So they are rates of consumption and/or income. The discrete streams model lump sum consumption or income. As both are deterministic they can be seen as already fixed future demands or investment schemes of the investor. Further, we assume that the investor can trade in a bond and a stock with prices given by

$$\begin{aligned} dP_0(t) &= P_0(t) r\, dt,\ P_0(0) = 1\,, \\ dP_1(t) &= P_1(t)\left[b\, dt + \sigma dW(t)\right],\ P_1(0) = p_1\,, \end{aligned}$$

with r, b, σ real constants, $\sigma > 0$ and $\{W(t), \mathcal{F}_t\}_{t\in[0,T]}$ a one-dimensional Brownian motion. In the presence of a continuous monetary stream c(t) and a discrete monetary stream D_i as above, the wealth process $X^\pi(t)$ of an investor who follows a portfolio process $\pi(t)$ is generated in the following way: On (t_i, t_{i+1}) it is given by

$$dX^\pi(t) = [X^\pi(t)r + X^\pi(t)(b-r)\pi(t) + c(t)]dt + X^\pi(t)\sigma\pi(t)dW_t \tag{1}$$

for $i = 0, \ldots, n$, with $t_0 = 0, t_{n+1} = T$. At time instants $t_1, \ldots, t_n$ we have the following jump condition

$$X^\pi(t_i) = X^\pi(t_i-) + D_i\,. \tag{2}$$

Our goal is to maximise the power utililty of the final wealth, i.e.

$$v(t,x) = \sup_{\pi\in\mathcal{A}(t,x)} E^{t,x}\left[\frac{1}{\gamma}X^\pi(T)^\gamma\right]\,, \tag{3}$$

where $\mathcal{A}(t,x)$ is the set of admissible portfolio processes, i.e. for all progressively measurable processes $\pi \in \mathcal{A}(t,x)$ the wealth process (1,2) with initial condition $X^{\pi}(t) = x$ has a solution $\{X^{\pi}(s)\}_{t\in[t,T]}$, and meets the conditions

$$E\left(\int_t^T |\pi(s)|^k ds\right) < \infty,$$

$$E\left(\sup_{s\in[t,T]} |X^{\pi}(s)|^k ds\right) < \infty$$

for all $k \in \mathbb{N}$, and $X^{\pi}(s) \geq 0$ for all $s \in [t,T]$ (For a detailed derivation of the wealth equation see Korn and Korn (2001)).

3 Constant Consumption Stream

As a first task we consider the case of a constant consumption process, i.e. we assume

$$c(t) \equiv -c \tag{4}$$

for all $t \in [0,T]$ and some $c > 0$. The HJB-equation corresponding to our problem (3) then has the form

$$\max_{\pi}\left\{{}^1\!/\!_2\pi^2\sigma^2x^2v_{xx}(t,x) + (rx + \pi(b-r)x - c)\,v_x(t,x) + v_t(t,x)\right\} = 0 \tag{5}$$

with the obvious final condition

$$v(T,x) = {}^1\!/\!_\gamma x^\gamma \tag{6}$$

and an appropriate boundary condition derived below. Ignoring this condition for the moment we note that a naive separation approach of the form

$$v(t,x) = {}^1\!/\!_\gamma x^\gamma f(t)$$

(see e.g.Korn (1997)) will not yield the explicit solution of (5-6). In fact, the constant term c in the HJB-equation is the reason for this approach not going through. The basic simple trick to solve our problem however, relies on the idea to devide the investor's initial capital x into a part needed to cover the payout stream for consumption and a remaining part which can be used for investing in the financial market. We first determine the minimal initial capital x_1 needed to finance the consumption stream via pure bond investment. The corresponding wealth process $Y(t)$ of future obligations obeys the following integral

$$Y(t) = \int_t^T \exp(-r(t-s))(-c)ds,$$

which equals

$$Y(t) = \tfrac{-c}{r}\left(1 - \exp\left(-r(T-t)\right)\right)$$

and determines x_1 via

$$x_1 = -Y(0) = \tfrac{c}{r}\left(1 - \exp(-rT)\right).$$

Subtracting this value from our initial capital x and investing the remaining amount according to the optimal portfolio process of the pure optimal terminal wealth problem which equals

$$\tilde{\pi}(t) = \frac{b-r}{\sigma^2(1-\gamma)}$$

(see Korn (1997)) leads to the following guess for the optimal wealth process

$$\begin{aligned} X(t) &= \left(x - \tfrac{c}{r}\left(1-\exp(-rT)\right)\right)\exp\left[\left(r + \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma} - {}^1\!/_2\left(\tfrac{b-r}{\sigma(1-\gamma)}\right)^2\right)t \right. \\ &\quad \left. + \frac{b-r}{\sigma(1-\gamma)}W(t)\right] + \tfrac{c}{r}\left(1-\exp\left(-r(T-t)\right)\right) \end{aligned} \tag{7}$$

with an expected final utility of

$$E\left(\tfrac{1}{\gamma}X(T)^\gamma\right) = \tfrac{1}{\gamma}\left(x - \tfrac{c}{r}\left(1-\exp(-rT)\right)\right)^\gamma \exp\left(\gamma\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)T\right).$$

Note in particular that for a given initial wealth smaller than x_1 the constant consumption rate process c cannot be financed at all. This is also true as a boundary in time. The investor's wealth is never allowed to drop below $-Y(t)$ to ensure that all the necessary future consumption payments can be covered. Thus, starting from an initial capital bigger than x the investor has to stop all his risky activites at time t if his wealth reaches

$$\tfrac{c}{r}\left(1-\exp\left(-r(T-t)\right)\right).$$

This fact adds the additional boundary condition

$$v\left(t, \tfrac{c}{r}\left(1-\exp\left(-r(T-t)\right)\right)\right) = 0 \tag{8}$$

for all $t \in [0,T]$ to the HJB-equation. Observe, that the boundary condition (8) ensures, that the wealth is always greater or equal to zero. Verifying the HJB-equation (5-6) together with this boundary condition proves optimality of the above described strategy:

Theorem 1 Optimal control with continuous consumption
Let our initial capital x satisfy

$$x > c\frac{1-\exp(-rT)}{r}. \tag{9}$$

Then, the value function v(t, x) of our optimisation problem (1-3) with a given consumption rate of $c \geq 0$ is given by

$$\begin{aligned} v(t,x) &= E^{t,x}\left(\tfrac{1}{\gamma}X(T)^\gamma\right) \\ &= \tfrac{1}{\gamma}\left(x - \tfrac{c}{r}\left(1-\exp\left(-r(T-t)\right)\right)\right)^\gamma \exp\left(\gamma\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)(T-t)\right) \end{aligned} \tag{10}$$

for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[\tfrac{c}{r}\left(1-\exp\left(-r(T-t)\right)\right), \infty\right)$.
The corresponding optimal portfolio process has the form

$$\pi(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[1 - \frac{c}{rx}\left(1-\exp\left(-r(T-t)\right)\right)\right]. \tag{11}$$

Remark
Note the following limiting behaviour of the portfolio process

$$\pi(t) \rightarrow \begin{cases} 0, \text{if} x \downarrow \frac{c}{r}\left(1-\exp\left(-r\left(T-t\right)\right)\right) \\ \frac{b-r}{\sigma^2(1-\gamma)}, \text{if} x \rightarrow \infty \end{cases}$$

I.e. the influence of the consumption vanishes if the wealth process approaches infinity while the consumption requirements do not permit stock investment if all the capital is needed for consumption. In particular, the boundary condition (8) is met.

Proof of Theorem 1
Standard verification theorems (see e.g. Fleming and Soner (1993) or Korn and Korn (2001)) yield that a smooth and polynomially bounded solution $v(t,x)$ of the HJB-equation (5-6) is indeed the value function of our optimisation problem. In doing the first step to arrive at this solution, we perform the optimisation in (5-6) which results in the candidate

$$\pi(t) = -\frac{b-r}{\sigma^2}\frac{v_x}{xv_{xx}}$$

for the optimal portfolio process and hence leads to the equation

$$v_t + (rx-c)\, v_x - {}^1\!/_2 \left(\frac{b-r}{\sigma}\right)^2 \frac{v_x^2}{v_{xx}} = 0\,,$$

which has to hold for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[\frac{c}{r}\left(1-\exp\left(-r\left(T-t\right)\right)\right), \infty\right)$ as points outside this set cannot guarantee to satisfy the consumption requirements for sure. We now verify that $v(t,x)$ as given in (10) solves this equation. To make this easier we introduce

$$A = \left(x - c\frac{1-\exp\left(-r\left(T-t\right)\right)}{r}\right), \quad B = \exp\left(\gamma\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)(T-t)\right).$$

and for the moment omit noting their dependence on x and t. Via $v = \frac{1}{\gamma}A^\gamma B$ this leads to

$$\begin{aligned} v_t &= A^{\gamma-1} c\exp\left(-r\left(T-t\right)\right) B - A^\gamma B\left(r + {}^1\!/_2\tfrac{1}{1-\gamma}\left(\tfrac{b-r}{\sigma}\right)^2\right) \\ v_x &= A^{\gamma-1}B \\ v_{xx} &= (\gamma-1)\, A^{\gamma-2}B \end{aligned}$$

and to

$$\begin{aligned} & v_t + (rx-c)\, v_x - {}^1\!/_2\left(\frac{b-r}{\sigma}\right)^2\frac{v_x^2}{v_{xx}} \\ = \;& (rx-c)\, A^{\gamma-1}B + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}A^\gamma B + A^{\gamma-1}c\exp\left(-r\left(T-t\right)\right)B \\ & -A^\gamma B\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right) \\ = \;& A^{\gamma-1}B\left[(rx-c) + c\exp\left(-r\left(T-t\right)\right) - Ar\right] = 0. \end{aligned}$$

As a further result we obtain the optimal portfolio process as

$$\pi(t) = -\frac{b-r}{\sigma^2}\frac{v_x}{xv_{xx}} = \frac{b-r}{\sigma^2(1-\gamma)}\left(1-\tfrac{c}{rx}\left(1-\exp\left(-r\left(T-t\right)\right)\right)\right).$$

As this process is bounded on the admissible set for (t,x) the corresponding wealth equation has a unique solution and also all requirements on an admissible control are met. □

Remark
Note that the form of the optimal portfolio process corresponds exactly to the strategy of dividing the initial capital into

$$x = x_1 + x_2 = c\frac{1-\exp(-rT)}{r} + (x - x_1)$$

and then leaving x_1 in the bond to pay out all the consumption requirements, taking the remaining part x_2 and investing it so as to solve a portfolio problem without any consumption at all. The second term in the brackets of the relation (11) defining $\pi(t)$ is thus a consequence of the consumption requirements. It can easily be verified that $\pi(t)$ satisfies all the integrability requirements of a portfolio process. The amount

$$c\frac{1-\exp\left(-r\left(T-t\right)\right)}{r}$$

has to be withdrawn from stock investment to make sure that the required consumption is safe.

4 Lump sum consumption

In contrast to the previous section we now assume that consumption takes place at fixed time instants $t_1, \ldots, t_n$ with $0 < t_1 < \ldots < t_n \leq T$ and is required to equal amounts $C_i > 0$ at times t_i. This is now a consumption stream with all mass concentrated at isolated points. However, the idea to put aside at $t = 0$ the required money to satisfy the needs for consumption and to invest the remaining capital as if there were no consumption at all, will stay valid here, too. Note that for paying in a consumption of $D_i = -C_i < 0$ at time t_i (i.e. to pay out $C_i > 0$)one needs an amount of money of

$$C_i e^{-r(t_i - t)}$$

at time $t \leq t_i$ to attain C_i via riskless investment on $[t, t_i]$. We therefore get the following condition for the wealth process to satisfy

$$X(t) \geq \sum_{i:t_i > t} C_i e^{-r(t_i - t)} \quad , \quad t \in [0, T]. \tag{12}$$

Note that by the form of this requirement we also indicate that $X(t)$ is the wealth at time t *after* the possible consumption at time t has been made. More precisely, we have

$$X(t_i) = X(t_i-) - C_i. \tag{13}$$

As $X(t)$ is discontinuous at the times of consumption t_i, we cannot expect the value function

$$v(t,x) = \sup_{\pi \in \mathcal{A}(t,x)} E^{t,x}\left[\frac{1}{\gamma}X^{\pi}(T)^{\gamma}\right] \tag{14}$$

to be continuous at t_i. Instead, we must have

$$v(t_i, x - C_i) = v(t_i-, x) \tag{15}$$

for all x satisfying (13) in place of $X(t)$. However, on intervalls (t_i, t_{i+1}) $v(t,x)$ should satisfy the usual HJB-Equation as we will prove in the verification theorem below. We summarize our consideration in

Theorem 2 Optimal control with lump sum consumption

For a given set of consumption requirements $C_i > 0$ *at times* t_i, $i = 0, \ldots, n$ *with* $0 \leq t_1 < \ldots < t_n \leq T$, *let our initial capital satisfy*

$$x > \sum_{i:t_i>t} C_i e^{-r(t_i-t)}\,. \tag{16}$$

Then the value function of problem (14) is given by

$$v(t,x) = \frac{1}{\gamma}\left(x - \sum_{i:t_i>t} C_i e^{-r(t_i-t)}\right)^{\gamma} e^{\gamma\left(r+\frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2\frac{1}{1-\gamma}\right)(T-t)} \tag{17}$$

for all pairs (t,x) with $t \in [0,T]$ *and* $x \in \left[\sum_{i:t_i>t} C_i e^{-r(t_i-t)}, \infty\right)$. *The corresponding optimal portfolio process has the form*

$$\pi(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[1 - \frac{\sum_{i:t_i>t} C_i e^{-r(t_i-t)}}{x}\right]. \tag{18}$$

Proof of Theorem 2
The verification theorem 3 below indicates that $v(t,x)$ is the unique (piecewise) smooth solution of the corresponding HJB equation that also satisfies the jump condition (15). Similar as in the proof of Theorem 1, we can verify that $v(t,x)$ as given in (17) above has these properties and hence coincides with the value function. One can also obtain the optimal portfolio process then directly as

$$\pi(t) = -\frac{b-r}{\sigma^2}\frac{v_x(t,x)}{xv_{xx}(t,x)} = \frac{b-r}{\sigma^2(1-\gamma)}\left[1 - \frac{\sum_{i:t_i>t} C_i e^{-r(t_i-t)}}{x}\right]$$

where at times t_i we have taken the right-continuous limit of the derivatives. □

It thus only remains to prove the verification theorem:

Theorem 3 Verification theorem for lump sum consumption
Let $g(t,x)$ *be a polynomially bounded solution of*

$$\sup_{\pi\in[-\alpha,\alpha]} \left\{\frac{1}{2}\sigma^2\pi^2x^2 g_{xx}(t,x) + x[r + \pi(b-r)]g_x(t,x) + g_t(t,x)\right\} = 0 \tag{19}$$

for all $t \in [0,T] \setminus \{t_1, \ldots, t_n\}, x > \sum_{i:t_i>t} c_i e^{-r(t_i-t)}$ *and some fixed* $\alpha > 0$.

$$g(t_i, x - C_i) = g(t_i-, x) \tag{20}$$

$$g\left(t, \sum_{i:t_i>t} C_i e^{-r(t_i-t)}\right) = 0 \tag{21}$$

$$g(T,x) = \frac{1}{\gamma}x^{\gamma} \tag{22}$$

which is in $C^{1,2}$ *on* $(t_i, t_{i+1}), i = 0, \ldots, n$ *with* $t_0 = 0, t_{n+1} = T$*. Le further be*

$$\pi^*(t,x) = -\frac{b-r}{\sigma^2}\frac{g_x(t,x)}{x g_{xx}(t,x)} \in (-\alpha, \alpha)$$

(where in points t_i *we take the right hand limits of the derivatives). Then,* $g(t,x)$ *coincides with the value function* $v(t,x)$ *, and* $\pi^*(t, X^{\pi^*}(t))$ *is an optimal portfolio process.*

Proof of Theorem 3

Let $g(t,x)$ be the asserted solution of (19)-(22). Let $\pi(.)$ be a portfolio process with corresponding wealth process $X^\pi(t)$ satisfying the initial condition (12) and $\pi(t) \in [-\alpha, \alpha]$. We then have:

$$\begin{aligned} g(t, X^\pi(t)) &= g(t_{i^e}, X^\pi(t_{i^e})) + \int_{t_{i^e}}^t \left[g_t + g_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 g_{xx} \right] ds \\ &\quad + \int_{t_{i^e}}^t g_x X^\pi(s) \sigma \pi(s) dW(s) \end{aligned}$$

for $i^e = \max\{i | t_i \leq t\}$. From (13) and (20) we have

$$g(t_{i^e}, X^\pi(t_{i^e})) = g(t_{i^e}-, X^\pi(t_{i^e}) + C_i) = g(t_{i^e}-, X^\pi(t_{i^e}-)) \quad .$$

Thus, starting at (t_s, x) we can apply the Itô-formula to obtain inductively

$$\begin{aligned} g(t, X^\pi(t)) = g(t_s, x) &+ \int_{t_s}^{t_{i^b}} \left[g_t + g_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 g_{xx} \right] ds \\ &+ \int_{t_s}^{t_{i^b}} g_x X^\pi(s) \sigma \pi(s) dW(s) \\ &+ \sum_{i=i^b}^{i^e-1} \int_{t_i}^{t_{i+1}} \left[g_t + g_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 g_{xx} \right] ds \\ &+ \sum_{i=i^b}^{i^e-1} \int_{t_i}^{t_{i+1}} g_x X^\pi(s) \sigma \pi(s) dW(s) \\ &+ \int_{t_{i^e}}^t \left[g_t + g_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 g_{xx} \right] ds \\ &+ \int_{t_{i^e}}^t g_x X^\pi(s) \sigma \pi(s) dW(s) \end{aligned}$$

where $i^b = \min\{i : t_i > t_s\}$. Due to the definition of $g(t,x)$ in (19), the fact that $\pi^*(t)$ attains the supremum in (19) and lies in $[-\alpha, \alpha]$ we have

$$E^{t_s,x}\left(g(t, X^\pi(t)) \leq E^{t_s,x}\left(g(t, X^{\pi^*})\right)\right.$$

for all $t \in [0,T]$ and $\pi \in \mathcal{A}(t_s, x)$ (note the polynomiality of $g(t,x)$ and the boundedness of $\pi(.)$, and $\pi^*(t)$) and in particular

$$E^{t_s,x}\left(\frac{1}{\gamma}\left(X^\pi(T)\right)^\gamma\right) = E^{t_s,x}\left(g(T, X^\pi(T)) \leq E^{t_s,x}\left(g(T, X^{\pi^*}(T))\right)\right. = E^{t_s,x}\left(\frac{1}{\gamma}\left(X^{\pi^*}(T)\right)^\gamma\right)$$

As $\pi^*(t) \in (-\alpha, \alpha)$, $\pi^*(t)$ is an (interior) optimal control, which is still optimal if we make α arbitrarily large. Hence $v(t,x) = E^{t_s,x}\left(g(T, X^{\pi^*}(T))\right)$, and using (19) we get

$$v(t,x) = E^{t_s,x}\left(g(T, X^{\pi^*}(T))\right) = g(t,x)$$

□

5 Generalized consumption and income

In the following we investigate the portolio problem with both consumption and income simultaneously. In both cases we deal with continuous and discrete monetary streams. More precisely, we assume that discrete consumption and income takes place at fixed time instants $t_1, \ldots, t_n$ with $0 < t_1 < \ldots < t_n \leq T$ and is required to equal values D_i at times t_i, where $D_i > 0$ means income and $D_i < 0$ means consumption. We denote the continuous monetary stream by $c(t)$, where again $c(t) < 0$ stands consumption and $c(t) > 0$ for income. Having seen both the relevant idea and the solution of the HJB-equation in sections 3 und 4, it is easy to figure out the necessary ingredients to solve the problem in the generalized case. Of course, if the value of future obligations is positive, we then do not have to set aside capital at the beginning. Just the opposite, as we are certain to get more capital in the future we can already take advantage of it. More precisely, we raise a credit to invest future income today to get a higher overall-return.

Observe, that the sign of the present value of future consumption and income can be changing over time. The main idea now is to add this present value - independent of its sign - to our wealth and to invest this then obtained capital as if there were no consumption or income at all.

The value of discrete streams D_i with $t_i > t$ equals

$$\sum_{i:t_i>t} D_i e^{-r(t_i-t)}.$$

The value of the continuous monetary stream $c(s)$ at time t equals

$$Y(t) = \int_t^T \exp(-r(s-t))c(s)ds.$$

In total, we get the following condition on the wealth process

$$X(t) \geq -Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \quad , \quad t \in [0,T]. \tag{23}$$

We solve this optimisation problem by using $u(t)$, the amount of money invested in the stocks as control process, instead of $\pi(t)$. The wealth process then has the representation

$$dX(t) = [X(t)r + (b-r)u(t) + c(t)]dt + u(t)\sigma dW_t \tag{24}$$

on (t_i, t_{i+1}) and the jump condition equals

$$X(t_i) = X(t_i-) + D_i. \tag{25}$$

Note that by the form of this requirement we also indicate that $X(t)$ is the wealth at time t after the discrete payment at time t has been made. We get the following value function

$$v(t,x) = \sup_{u \in \mathcal{A}^*(t,x)} E^{t,x}\left[\frac{1}{\gamma}X^u(T)^\gamma\right] \tag{26}$$

with the obvious jump condition

$$v(t_i, x + D_i) = v(t_i-, x) \tag{27}$$

for all x satisfying (23) in place of $X(t)$ and $\mathcal{A}^*(t,x)$ the corresponding admissible set of controls for $u(t)$. However, on intervalls (t_i, t_{i+1}) $v(t,x)$ should satisfy the usual HJB-Equation as we will prove in the verification theorem below.

Remark
The corresponding boundary condition of the value function is

$$v\left(t, -Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right) = 0\,.$$

So if the value of future streams is positive at a particular time instant $\tilde{t} \in [0,T]$, the domain of $v(t,x)$ and the corresponding control includes points (t,x) with x=0 (in particular $(\tilde{t},0)$). So looking at the optimal controls (11) and (18) we see that just copying the methods of section 3 or 4 cannot work, since π would not be defined for x=0. We therefore overcome this problem by choosing as control $u(t)$, the process of money invested in the stock instead of the portfolio process $\pi(t)$. However, the main ideas will stay valid here.

Theorem 4 Optimisation with consumption and income
For a given set of discrete streams D_i at times t_i, $i = 1,\ldots,n$,with $0 \le t_1 < \ldots < t_n \le T$ and a continuous stream $c(s)$ with present value $Y(t) = \int_t^T \exp(-r(t-s))c(s)ds$, let our initial capital satisfy

$$x > -Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}. \tag{28}$$

Then the value function of problem (26) is given by

$$v(t,x) = \frac{1}{\gamma}\left(x + Y(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right)^\gamma e^{\gamma\left(r+\frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2\frac{1}{1-\gamma}\right)(T-t)} \tag{29}$$

for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[-Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}, \infty\right)$. The corresponding process of amounts of money invested in the stock has the form

$$u^*(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[X(t) + Y(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right].$$

Proof of Theorem 4
The HJB-equation corresponding to our problem has the form

$$\sup_{u\in I\!R}\left\{{}^1\!/_2 u^2\sigma^2 v_{xx}\left(t,x\right)+\left(rx+\left(b-r\right)u+c(t)\right)v_x\left(t,x\right)+v_t\left(t,x\right)\right\}=0 \tag{30}$$

for all pairs (t,x) with $t\in[0,T]\setminus\{t_1,\ldots,t_n\}$ and $t\in\left(-Y(t)-\sum_{i:t_i>t}D_i e^{-r(t_i-t)},\infty\right)$ and boundary conditions

$$\begin{aligned} v\left(T,x\right) &= {}^1\!/_\gamma x^\gamma \\ v\left(t,-Y(t)-\sum_{i:t_i>t}D_i e^{-r(t_i-t)}\right) &= 0 \\ v(t_i,x+D_i) &= v(t_i-,x) \end{aligned}$$

The verification theorem 5 below indicates that $v(t,x)$ is the unique (piecewise) smooth solution of the corresponding HJB equation that also satisfies the jump condition (27). We will verify that $v(t,x)$ as given in (29) above has these properties and hence coincides with the value function. In doing the first step to arrive at this solution we perform the optimisation in (30) which results in the candidate

$$u(t)=-\frac{b-r}{\sigma^2}\frac{v_x(t,x)}{v_{xx}(t,x)}\,,$$

where at times t_i we have taken the right-continuous limit of the derivatives. For the optimal portfolio process and as a consequence this leads to the equation

$$v_t+\left(rx+c(t)\right)v_x-{}^1\!/_2\left(\frac{b-r}{\sigma}\right)^2\frac{v_x^2}{v_{xx}}=0\,,$$

which has the same domain as the HJB-equation (30). We now verify that $v(t,x)$ as given in (29) solves this equation. To make this easier we introduce

$$A=x+Y(t)+\sum_{i:t_i>t}D_i e^{-r(t_i-t)},\quad B=\exp\left(\gamma\left(r+{}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)(T-t)\right).$$

This leads to

$$\begin{aligned} v_t &= A^{\gamma-1}\left(Y'(t)+r\sum_{i:t_i>t}D_i e^{-r(t_i-t)}\right)B-A^\gamma B\left(r+{}^1\!/_2\tfrac{1}{1-\gamma}\left(\tfrac{b-r}{\sigma}\right)^2\right) \\ v_x &= A^{\gamma-1}B \\ v_{xx} &= (\gamma-1)\,A^{\gamma-2}B \end{aligned}$$

and to

$$
\begin{aligned}
& v_t + (rx + c(t))\, v_x - 1/2 \left(\frac{b-r}{\sigma}\right)^2 \frac{v_x^2}{v_{xx}} \\
= \; & (rx + c(t))\, A^{\gamma-1} B + 1/2 \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma} A^\gamma B + A^{\gamma-1} \left(Y'(t) + r \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \right) B \\
& - A^\gamma B \left(r + 1/2 \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma} \right) \\
= \; & A^{\gamma-1} B \left[(rx + c(t)) + \left(Y'(t) + r \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \right) - Ar \right] \\
= \; & A^{\gamma-1} B \left[c(t) + Y'(t) - Y(t) \right] = 0.
\end{aligned}
$$

As a further result we obtain the optimal portfolio process

$$
u^*(t) = -\frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}} = \frac{b-r}{\sigma^2 (1-\gamma)} \left(X(t) + Y(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \right).
$$

where at times t_i we have taken the right-continuous limit of the derivatives.

□

It thus only remains to prove the verification theorem.

Theorem 5 Verification theorem
Let g(t,x) be a polynomially bounded solution of

$$
\sup_{\pi \in I\!R} \{ \frac{1}{2} \sigma^2 u^2 g_{xx}(t,x) + [xr + (b-r)u] g_x(t,x) + g_t(t,x) \} = 0 \tag{31}
$$

for all $t \in [0,T] \setminus \{t_1, \ldots, t_n\}, x > -Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}$ *and*

$$
g(t_i-, x) = g(t_i, x + D_i) \tag{32}
$$

$$
g\left(t, -Y(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \right) = 0 \tag{33}
$$

$$
g(T, x) = \frac{1}{\gamma} x^\gamma \tag{34}
$$

which is in $C^{1,2}$ *on* $(t_i, t_{i+1}), i = 0, \ldots, n$ *with* $t_0 = 0, t_{n+1} = T$. *Le further be*

$$
u^*(t,x) = -\frac{b-r}{\sigma^2} \frac{g_x(t,x)}{g_{xx}(t,x)}
$$

(where in points t_i *we take the right hand limits of the derivatives). Then,* $g(t,x)$ *coincides with the value function* $v(t,x)$*, and* $u^*(t, X^{u^*}(t))$ *is an optimal control process for problem (26).*

Proof of Theorem 5
Let $g(t,x)$ be the asserted solution of (31-34). Let $u(.)$ be a portfolio process with corresponding wealth process $X^u(t)$ satisfying (23). Again, we apply Itô formula to obtain inductively for each admissible control process $u(.)$

$$\begin{aligned} g(t,X^\pi(t)) = g(t_s,x) \quad &+ \int_{t_s}^{t_{i^b}} \left[g_t + g_x(X^u(s)r + u(s)(b-r) + c(s)) + \frac{1}{2}\sigma^2 u(s)^2 g_{xx}\right] ds \\ &+ \int_{t_s}^{t_{i^b}} g_x \sigma u(s) dW(s) \\ &+ \sum_{i=i^b}^{i^e} \int_{t_i}^{t_{i+1}} \left[g_t + g_x(X^u(s)r + u(s)(b-r) + c(s)) + \frac{1}{2}\sigma^2 u(s)^2 g_{xx}\right] ds \\ &+ \sum_{i=i^b}^{i^e} \int_{t_i}^{t_{i+1}} g_x \sigma u(s) dW(s) \\ &+ \int_{t_{i^e}}^{t} \left[g_t + g_x(X^u(s)r + u(s)(b-r) + c(s)) + \frac{1}{2}\sigma^2 u(s)^2 g_{xx}\right] ds \\ &+ \int_{t_{i^e}}^{t} g_x \sigma u(s) dW(s) \end{aligned}$$

where $i^b = \min\{i : t_i > t_s\}$ and $i^e = \max\{i : t_i \leq t\}$. Due to the form of $u^*(t,x)$ (an affine linear function of $X^*(t)$), $X^*(t)$ is the unique solution of the corresponding wealth equation . Furter as $g(t,x)$ solves (31) and the fact that $u^*(t)$ attains the supremum in (31) we have:

$$E^{t_s,x}\left(\frac{1}{\gamma}\left(X^u(T)\right)^\gamma\right) = E^{t_s,x}\left(g(T,X^u(T)\right) \leq E^{t_s,x}\left(g(T,X^{u^*}(T))\right) = E^{t_s,x}\left(\frac{1}{\gamma}\left(X^{u^*}(T)\right)^\gamma\right)$$

where the expectations are finite due to the polynomiality of $g(t,x)$. Hence:

$$v(t,x) = E^{t_s,x}\left(g(T,X^{\pi^*}(T))\right) = g(t,x).$$

□

6 Numerical illustration and conclusions

To illustrate the behaviour of the portfolio process in the different situations presented so far we give some numerical examples:

Figure 1 corresponds to the continuous consumption case in section 3 with $T = 1$, $b = 12\%$, $r = 5\%$, $\sigma = 20\%$, $\gamma = 0.5$ and consumption rate $c(t) \equiv -500$. The optimal control without consumption would be $\pi' = \frac{b-r}{(1-\gamma)\sigma^2} = 3.5$. We see that for $t \to 1$ and x constant the optimal control $\pi(t,x)$ converges to 3.5, since the amount of consumption, which has to be financed from the wealth is decreasing with time, and so we have more and more money left over to invest in stocks. For increasing wealth x and constant t the optimal control converges again to 3.5, since the role of consumption compared with total wealth can then be neglected. On the other hand, for $x \to c\frac{1-\exp(-r(T-t))}{r}$, $\pi(t,x)$ converges to zero, since if $X(t) = c\frac{1-\exp(-r(T-t))}{r}$, all wealth is needed to finance future consumption.

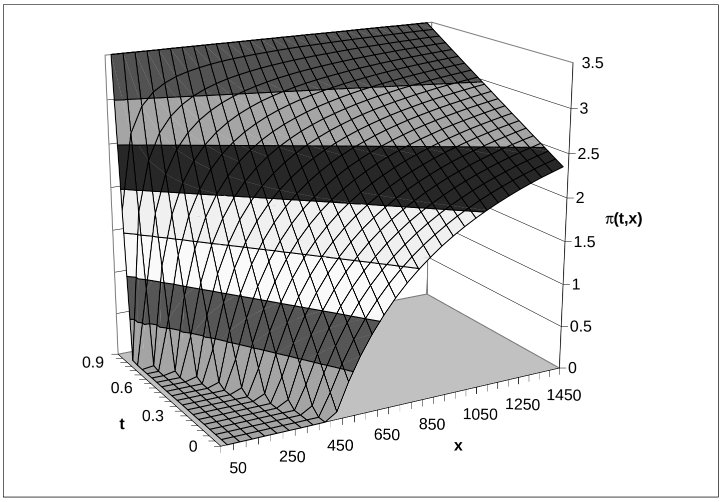

Figure 1: Optimal control with continuous consumption

Figure 2 shows the optimal control for the discrete consumption case in section 4 with same stock parameters as above but lump sum consumption with $\triangle t = 0.2$ and $D_i \equiv -100$. It is not surprising, that we get jumps at consumption time instants. Besides this effect, the behaviour coincides with that of Figure 1.

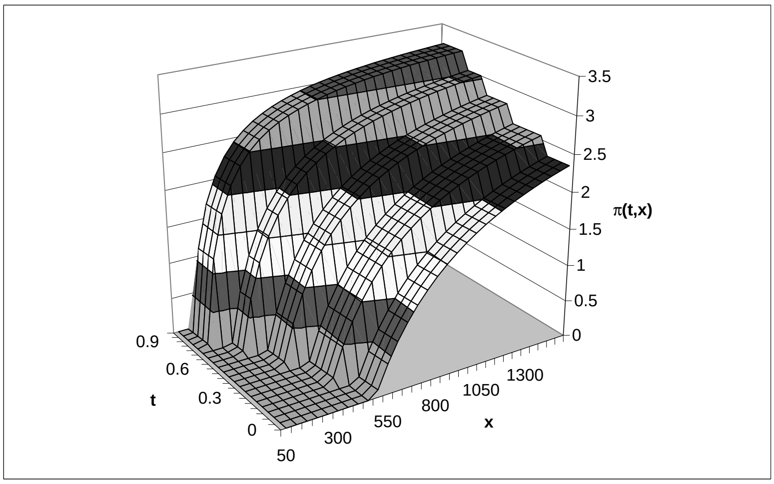

Figure 2: Optimal control with lump sum consumption

Figure 3 and Figure 4 illustrate the difference between continuous income and consumption, where we used the same parameters as before, except $c(t) = +500$ for the income rate. Note, that we changed the control process to be the amount of money invested in the stock instead of the portfolio process. In the case of income the optimal control decreases over time, because the amount of future income decreases. In the case for consumption it is just the other way around, i.e. the optimal control increases, since the money needed to finance future consumption decreases.

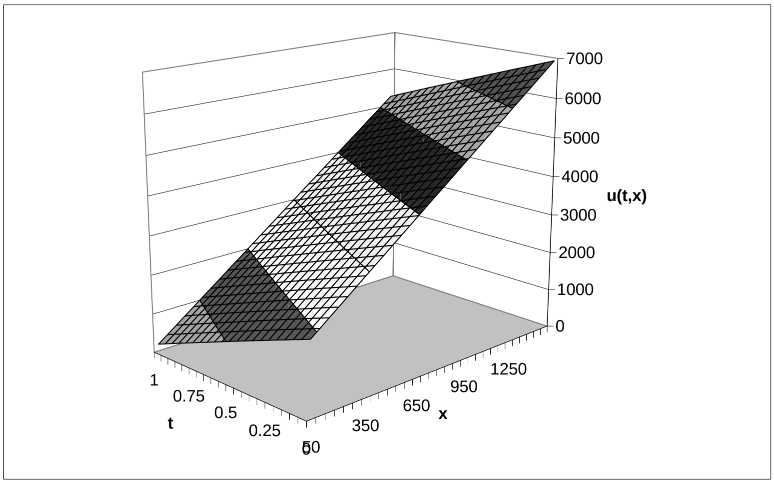

Figure 3: Optimal control with continuous income

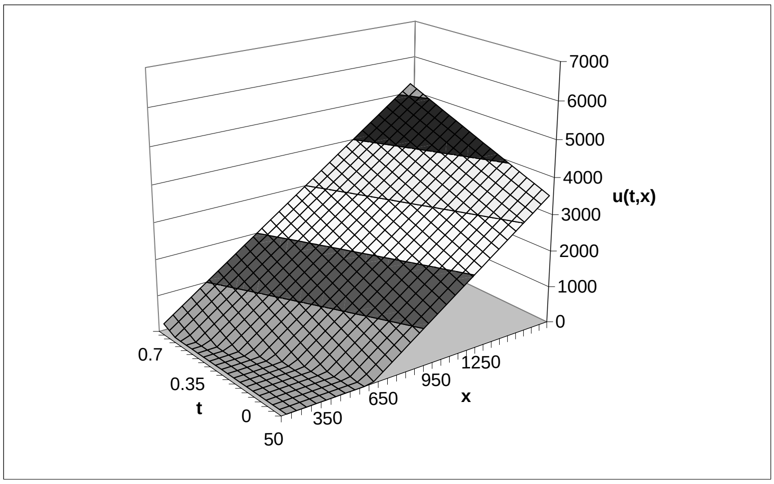

Figure 4: Optimal control with continuous consumption

Conclusions

As private equity plans on one hand are getting more and more into fashion we believe that the results of this paper have a practical relevance. Further, the case of an a priori fixed consumption plan seems to be much more realistic than that of a random consumption as treated in the standard formulation of the portfolio problem. With regard to this argument and our results one can thus always concentrate on the pure terminal wealth problem.

Even more general problems can be treated with our approach and are subjects of future research. Two possible candidates are: Optimal portfolios with fixed consumption/income and a loan depending interest rate (see Krekel (2001) for a related problem) and optimal portfolios with crash possibilities and fixed consumption/income (see Korn (2001)).

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1. D. Hietel, K. Steiner, J. Struckmeier

A Finite - Volume Particle Method for Compressible Flows

We derive a new class of particle methods for conservation laws, which are based on numerical flux functions to model the interactions between moving particles. The derivation is similar to that of classical Finite-Volume methods; except that the fixed grid structure in the Finite-Volume method is substituted by so-called mass packets of particles. We give some numerical results on a shock wave solution for Burgers equation as well as the well-known one-dimensional shock tube problem.
(19 S., 1998)

2. M. Feldmann, S. Seibold

Damage Diagnosis of Rotors: Application of Hilbert Transform and Multi-Hypothesis Testing

In this paper, a combined approach to damage diagnosis of rotors is proposed. The intention is to employ signal-based as well as model-based procedures for an improved detection of size and location of the damage. In a first step, Hilbert transform signal processing techniques allow for a computation of the signal envelope and the instantaneous frequency, so that various types of non-linearities due to a damage may be identified and classified based on measured response data. In a second step, a multi-hypothesis bank of Kalman Filters is employed for the detection of the size and location of the damage based on the information of the type of damage provided by the results of the Hilbert transform.
Keywords:
Hilbert transform, damage diagnosis, Kalman filtering, non-linear dynamics
(23 S., 1998)

3. Y. Ben-Haim, S. Seibold

Robust Reliability of Diagnostic Multi-Hypothesis Algorithms: Application to Rotating Machinery

Damage diagnosis based on a bank of Kalman filters, each one conditioned on a specific hypothesized system condition, is a well recognized and powerful diagnostic tool. This multi-hypothesis approach can be applied to a wide range of damage conditions. In this paper, we will focus on the diagnosis of cracks in rotating machinery. The question we address is: how to optimize the multi-hypothesis algorithm with respect to the uncertainty of the spatial form and location of cracks and their resulting dynamic effects. First, we formulate a measure of the reliability of the diagnostic algorithm, and then we discuss modifications of the diagnostic algorithm for the maximization of the reliability. The reliability of a diagnostic algorithm is measured by the amount of uncertainty consistent with no-failure of the diagnosis. Uncertainty is quantitatively represented with convex models.
Keywords:
Robust reliability, convex models, Kalman filtering, multi-hypothesis diagnosis, rotating machinery, crack diagnosis
(24 S., 1998)

4. F.-Th. Lentes, N. Siedow

Three-dimensional Radiative Heat Transfer in Glass Cooling Processes

For the numerical simulation of 3D radiative heat transfer in glasses and glass melts, practically applicable mathematical methods are needed to handle such problems optimal using workstation class computers. Since the exact solution would require super-computer capabilities we concentrate on approximate solutions with a high degree of accuracy. The following approaches are studied: 3D diffusion approximations and 3D ray-tracing methods.
(23 S., 1998)

5. A. Klar, R. Wegener

A hierarchy of models for multilane vehicular traffic
Part I: Modeling

In the present paper multilane models for vehicular traffic are considered. A microscopic multilane model based on reaction thresholds is developed. Based on this model an Enskog like kinetic model is developed. In particular, care is taken to incorporate the correlations between the vehicles. From the kinetic model a fluid dynamic model is derived. The macroscopic coefficients are deduced from the underlying kinetic model. Numerical simulations are presented for all three levels of description in [10]. Moreover, a comparison of the results is given there.
(23 S., 1998)

Part II: Numerical and stochastic investigations

In this paper the work presented in [6] is continued. The present paper contains detailed numerical investigations of the models developed there. A numerical method to treat the kinetic equations obtained in [6] are presented and results of the simulations are shown. Moreover, the stochastic correlation model used in [6] is described and investigated in more detail.
(17 S., 1998)

6. A. Klar, N. Siedow

Boundary Layers and Domain Decomposition for Radiative Heat Transfer and Diffusion Equations: Applications to Glass Manufacturing Processes

In this paper domain decomposition methods for radiative transfer problems including conductive heat transfer are treated. The paper focuses on semi-transparent materials, like glass, and the associated conditions at the interface between the materials. Using asymptotic analysis we derive conditions for the coupling of the radiative transfer equations and a diffusion approximation. Several test cases are treated and a problem appearing in glass manufacturing processes is computed. The results clearly show the advantages of a domain decomposition approach. Accuracy equivalent to the solution of the global radiative transfer solution is achieved, whereas computation time is strongly reduced.
(24 S., 1998)

7. I. Choquet

Heterogeneous catalysis modelling and numerical simulation in rarified gas flows
Part I: Coverage locally at equilibrium

A new approach is proposed to model and simulate numerically heterogeneous catalysis in rarefied gas flows. It is developed to satisfy all together the following points:
1) describe the gas phase at the microscopic scale, as required in rarefied flows,
2) describe the wall at the macroscopic scale, to avoid prohibitive computational costs and consider not only crystalline but also amorphous surfaces,
3) reproduce on average macroscopic laws correlated with experimental results and
4) derive analytic models in a systematic and exact way.
The problem is stated in the general framework of a non static flow in the vicinity of a catalytic and non porous surface (without aging). It is shown that the exact and systematic resolution method based on the Laplace transform, introduced previously by the author to model collisions in the gas phase, can be extended to the present problem. The proposed approach is applied to the modelling of the Eley-Rideal and Langmuir-Hinshelwood recombinations, assuming that the coverage is locally at equilibrium. The models are developed considering one atomic species and extended to the general case of several atomic species. Numerical calculations show that the models derived in this way reproduce with accuracy behaviors observed experimentally.
(24 S., 1998)

8. J. Ohser, B. Steinbach, C. Lang

Efficient Texture Analysis of Binary Images

A new method of determining some characteristics of binary images is proposed based on a special linear filtering. This technique enables the estimation of the area fraction, the specific line length, and the specific integral of curvature. Furthermore, the specific length of the total projection is obtained, which gives detailed information about the texture of the image. The influence of lateral and directional resolution depending on the size of the applied filter mask is discussed in detail. The technique includes a method of increasing directional resolution for texture analysis while keeping lateral resolution as high as possible.
(17 S., 1998)

9. J. Orlik

Homogenization for viscoelasticity of the integral type with aging and shrinkage

A multi-phase composite with periodic distributed inclusions with a smooth boundary is considered in this contribution. The composite component materials are supposed to be linear viscoelastic and aging (of the non-convolution integral type, for which the Laplace transform with respect to time is not effectively applicable) and are subjected to isotropic shrinkage. The free shrinkage deformation can be considered as a fictitious temperature deformation in the behavior law. The procedure presented in this paper proposes a way to determine average (effective homogenized) viscoelastic and shrinkage (temperature) composite properties and the homogenized stress-field from known properties of the

components. This is done by the extension of the asymptotic homogenization technique known for pure elastic non-homogeneous bodies to the non-homogeneous thermo-viscoelasticity of the integral non-convolution type. Up to now, the homogenization theory has not covered viscoelasticity of the integral type.
Sanchez-Palencia (1980), Francfort & Suquet (1987) (see [2], [9]) have considered homogenization for viscoelasticity of the differential form and only up to the first derivative order. The integral-modeled viscoelasticity is more general then the differential one and includes almost all known differential models. The homogenization procedure is based on the construction of an asymptotic solution with respect to a period of the composite structure. This reduces the original problem to some auxiliary boundary value problems of elasticity and viscoelasticity on the unit periodic cell, of the same type as the original non-homogeneous problem. The existence and uniqueness results for such problems were obtained for kernels satisfying some constrain conditions. This is done by the extension of the Volterra integral operator theory to the Volterra operators with respect to the time, whose 1 kernels are space linear operators for any fixed time variables. Some ideas of such approach were proposed in [11] and [12], where the Volterra operators with kernels depending additionally on parameter were considered. This manuscript delivers results of the same nature for the case of the space-operator kernels.
(20 S., 1998)

10. J. Mohring

Helmholtz Resonators with Large Aperture

The lowest resonant frequency of a cavity resonator is usually approximated by the classical Helmholtz formula. However, if the opening is rather large and the front wall is narrow this formula is no longer valid. Here we present a correction which is of third order in the ratio of the diameters of aperture and cavity. In addition to the high accuracy it allows to estimate the damping due to radiation. The result is found by applying the method of matched asymptotic expansions. The correction contains form factors describing the shapes of opening and cavity. They are computed for a number of standard geometries. Results are compared with numerical computations.
(21 S., 1998)

11. H. W. Hamacher, A. Schöbel

On Center Cycles in Grid Graphs

Finding "good" cycles in graphs is a problem of great interest in graph theory as well as in locational analysis. We show that the center and median problems are NP hard in general graphs. This result holds both for the variable cardinality case (i.e. all cycles of the graph are considered) and the fixed cardinality case (i.e. only cycles with a given cardinality p are feasible). Hence it is of interest to investigate special cases where the problem is solvable in polynomial time.
In grid graphs, the variable cardinality case is, for instance, trivially solvable if the shape of the cycle can be chosen freely.
If the shape is fixed to be a rectangle one can analyze rectangles in grid graphs with, in sequence, fixed dimension, fixed cardinality, and variable cardinality. In all cases a complete characterization of the optimal cycles and closed form expressions of the optimal objective values are given, yielding polynomial time algorithms for all cases of center rectangle problems.
Finally, it is shown that center cycles can be chosen as rectangles for small cardinalities such that the center cycle problem in grid graphs is in these cases completely solved.
(15 S., 1998)

12. H. W. Hamacher, K.-H. Küfer

Inverse radiation therapy planning - a multiple objective optimisation approach

For some decades radiation therapy has been proved successful in cancer treatment. It is the major task of clinical radiation treatment planning to realize on the one hand a high level dose of radiation in the cancer tissue in order to obtain maximum tumor control. On the other hand it is obvious that it is absolutely necessary to keep in the tissue outside the tumor, particularly in organs at risk, the unavoidable radiation as low as possible.
No doubt, these two objectives of treatment planning - high level dose in the tumor, low radiation outside the tumor - have a basically contradictory nature. Therefore, it is no surprise that inverse mathematical models with dose distribution bounds tend to be infeasible in most cases. Thus, there is need for approximations compromising between overdosing the organs at risk and underdosing the target volume.
Differing from the currently used time consuming iterative approach, which measures deviation from an ideal (non-achievable) treatment plan using recursively trial-and-error weights for the organs of interest, we go a new way trying to avoid a priori weight choices and consider the treatment planning problem as a multiple objective linear programming problem: with each organ of interest, target tissue as well as organs at risk, we associate an objective function measuring the maximal deviation from the prescribed doses.
We build up a data base of relatively few efficient solutions representing and approximating the variety of Pareto solutions of the multiple objective linear programming problem. This data base can be easily scanned by physicians looking for an adequate treatment plan with the aid of an appropriate online tool.
(14 S., 1999)

13. C. Lang, J. Ohser, R. Hilfer

On the Analysis of Spatial Binary Images

This paper deals with the characterization of microscopically heterogeneous, but macroscopically homogeneous spatial structures. A new method is presented which is strictly based on integral-geometric formulae such as Crofton's intersection formulae and Hadwiger's recursive definition of the Euler number. The corresponding algorithms have clear advantages over other techniques. As an example of application we consider the analysis of spatial digital images produced by means of Computer Assisted Tomography.
(20 S., 1999)

14. M. Junk

On the Construction of Discrete Equilibrium Distributions for Kinetic Schemes

A general approach to the construction of discrete equilibrium distributions is presented. Such distribution functions can be used to set up Kinetic Schemes as well as Lattice Boltzmann methods. The general principles are also applied to the construction of Chapman Enskog distributions which are used in Kinetic Schemes for compressible Navier-Stokes equations.
(24 S., 1999)

15. M. Junk, S. V. Raghurame Rao

A new discrete velocity method for Navier-Stokes equations

The relation between the Lattice Boltzmann Method, which has recently become popular, and the Kinetic Schemes, which are routinely used in Computational Fluid Dynamics, is explored. A new discrete velocity model for the numerical solution of Navier-Stokes equations for incompressible fluid flow is presented by combining both the approaches. The new scheme can be interpreted as a pseudo-compressibility method and, for a particular choice of parameters, this interpretation carries over to the Lattice Boltzmann Method.
(20 S., 1999)

16. H. Neunzert

Mathematics as a Key to Key Technologies

The main part of this paper will consist of examples, how mathematics really helps to solve industrial problems; these examples are taken from our Institute for Industrial Mathematics, from research in the Technomathematics group at my university, but also from ECMI groups and a company called TecMath, which originated 10 years ago from my university group and has already a very successful history.
(39 S. (vier PDF-Files), 1999)

17. J. Ohser, K. Sandau

Considerations about the Estimation of the Size Distribution in Wicksell's Corpuscle Problem

Wicksell's corpuscle problem deals with the estimation of the size distribution of a population of particles, all having the same shape, using a lower dimensional sampling probe. This problem was originary formulated for particle systems occurring in life sciences but its solution is of actual and increasing interest in materials science. From a mathematical point of view, Wicksell's problem is an inverse problem where the interesting size distribution is the unknown part of a Volterra equation. The problem is often regarded ill-posed, because the structure of the integrand implies unstable numerical solutions. The accuracy of the numerical solutions is considered here using the condition number, which allows to compare different numerical methods with different (equidistant) class sizes and which indicates, as one result, that a finite section thickness of the probe reduces the numerical problems. Furthermore, the relative error of estimation is computed which can be split into two parts. One part consists of the relative discretization error that increases for increasing class size, and the second part is related to the relative statistical error which increases with decreasing class size. For both parts, upper bounds can be given and the sum of them indicates an optimal class width depending on some specific constants.
(18 S., 1999)

18. E. Carrizosa, H. W. Hamacher, R. Klein, S. Nickel

Solving nonconvex planar location problems by finite dominating sets

It is well-known that some of the classical location problems with polyhedral gauges can be solved in polynomial time by finding a finite dominating set, i. e. a finite set of candidates guaranteed to contain at least one optimal location.
In this paper it is first established that this result holds for a much larger class of problems than currently considered in the literature. The model for which this result can be proven includes, for instance, location problems with attraction and repulsion, and location-allocation problems.
Next, it is shown that the approximation of general gauges by polyhedral ones in the objective function of our general model can be analyzed with regard to the subsequent error in the optimal objective value. For the approximation problem two different approaches are described, the sandwich procedure and the greedy algorithm. Both of these approaches lead - for fixed epsilon - to polynomial approximation algorithms with accuracy epsilon for solving the general model considered in this paper.
Keywords:
Continuous Location, Polyhedral Gauges, Finite Dominating Sets, Approximation, Sandwich Algorithm, Greedy Algorithm
(19 S., 2000)

19. A. Becker

A Review on Image Distortion Measures

Within this paper we review image distortion measures. A distortion measure is a criterion that assigns a "quality number" to an image. We distinguish between mathematical distortion measures and those distortion measures in-cooperating a priori knowledge about the imaging devices (e. g. satellite images), image processing algorithms or the human physiology. We will consider representative examples of different kinds of distortion measures and are going to discuss them.
Keywords:
Distortion measure, human visual system
(26 S., 2000)

20. H. W. Hamacher, M. Labbé, S. Nickel, T. Sonneborn

Polyhedral Properties of the Uncapacit*ated Multiple Allocation Hub Location Problem*

We examine the feasibility polyhedron of the uncapacitated hub location problem (UHL) with multiple allocation, which has applications in the fields of air passenger and cargo transportation, telecommunication and postal delivery services. In particular we determine the dimension and derive some classes of facets of this polyhedron. We develop some general rules about lifting facets from the uncapacitated facility location (UFL) for UHL and projecting facets from UHL to UFL. By applying these rules we get a new class of facets for UHL which dominates the inequalities in the original formulation. Thus we get a new formulation of UHL whose constraints are all facet–defining. We show its superior computational performance by benchmarking it on a well known data set.
Keywords:
integer programming, hub location, facility location, valid inequalities, facets, branch and cut
(21 S., 2000)

21. H. W. Hamacher, A. Schöbel

Design of Zone Tariff Systems in Public Transportation

Given a public transportation system represented by its stops and direct connections between stops, we consider two problems dealing with the prices for the customers: The fare problem in which subsets of stops are already aggregated to zones and "good" tariffs have to be found in the existing zone system. Closed form solutions for the fare problem are presented for three objective functions. In the zone problem the design of the zones is part of the problem. This problem is NP hard and we therefore propose three heuristics which prove to be very successful in the redesign of one of Germany's transportation systems.
(30 S., 2001)

22. D. Hietel, M. Junk, R. Keck, D. Teleaga:

The Finite-Volume-Particle Method for Conservation Laws

In the Finite-Volume-Particle Method (FVPM), the weak formulation of a hyperbolic conservation law is discretized by restricting it to a discrete set of test functions. In contrast to the usual Finite-Volume approach, the test functions are not taken as characteristic functions of the control volumes in a spatial grid, but are chosen from a partition of unity with smooth and overlapping partition functions (the particles), which can even move along prescribed velocity fields. The information exchange between particles is based on standard numerical flux functions. Geometrical information, similar to the surface area of the cell faces in the Finite-Volume Method and the corresponding normal directions are given as integral quantities of the partition functions.
After a brief derivation of the Finite-Volume-Particle Method, this work focuses on the role of the geometric coefficients in the scheme.
(16 S., 2001)

23. T. Bender, H. Hennes, J. Kalcsics, M. T. Melo, S. Nickel

Location Software and Interface with GIS and Supply Chain Management

The objective of this paper is to bridge the gap between location theory and practice. To meet this objective focus is given to the development of software capable of addressing the different needs of a wide group of users. There is a very active community on location theory encompassing many research fields such as operations research, computer science, mathematics, engineering, geography, economics and marketing. As a result, people working on facility location problems have a very diverse background and also different needs regarding the software to solve these problems. For those interested in non-commercial applications (e. g. students and researchers), the library of location algorithms (LoLA can be of considerable assistance. LoLA contains a collection of efficient algorithms for solving planar, network and discrete facility location problems. In this paper, a detailed description of the functionality of LoLA is presented. In the fields of geography and marketing, for instance, solving facility location problems requires using large amounts of demographic data. Hence, members of these groups (e. g. urban planners and sales managers) often work with geographical information too s. To address the specific needs of these users, LoLA was inked to a geographical information system (GIS) and the details of the combined functionality are described in the paper. Finally, there is a wide group of practitioners who need to solve large problems and require special purpose software with a good data interface. Many of such users can be found, for example, in the area of supply chain management (SCM). Logistics activities involved in strategic SCM include, among others, facility location planning. In this paper, the development of a commercial location software tool is also described. The too is embedded in the Advanced Planner and Optimizer SCM software developed by SAP AG, Walldorf, Germany. The paper ends with some conclusions and an outlook to future activities.
Keywords:
facility location, software development, geographical information systems, supply chain management.
(48 S., 2001)

24. H. W. Hamacher, S. A. Tjandra

Mathematical Modelling of Evacuation Problems: A State of Art

This paper details models and algorithms which can be applied to evacuation problems. While it concentrates on building evacuation many of the results are applicable also to regional evacuation. All models consider the time as main parameter, where the travel time between components of the building is part of the input and the overall evacuation time is the output. The paper distinguishes between macroscopic and microscopic evacuation models both of which are able to capture the evacuees' movement over time.
Macroscopic models are mainly used to produce good lower bounds for the evacuation time and do not consider any individual behavior during the emergency situation. These bounds can be used to analyze existing buildings or help in the design phase of planning a building. Macroscopic approaches which are based on dynamic network flow models (minimum cost dynamic flow, maximum dynamic flow, universal maximum flow, quickest path and quickest flow) are described. A special feature of the presented approach is the fact, that travel times of evacuees are not restricted to be constant, but may be density dependent. Using multicriteria optimization priority regions and blockage due to fire or smoke may be considered. It is shown how the modelling can be done using time parameter either as discrete or continuous parameter.
Microscopic models are able to model the individual evacuee's characteristics and the interaction among evacuees which influence their movement. Due to the corresponding huge amount of data one uses simulation approaches. Some probabilistic laws for individual evacuee's movement are presented. Moreover ideas to model the evacuee's movement using cellular automata (CA) and resulting software are presented.
In this paper we will focus on macroscopic models and only summarize some of the results of the microscopic approach. While most of the results are applicable to general evacuation situations, we concentrate on building evacuation.
(44 S., 2001)

25. J. Kuhnert, S. Tiwari
Grid free method for solving the Poisson equation

A Grid free method for solving the Poisson equation is presented. This is an iterative method. The method is based on the weighted least squares approximation in which the Poisson equation is enforced to be satisfied in every iterations. The boundary conditions can also be enforced in the iteration process. This is a local approximation procedure. The Dirichlet, Neumann and mixed boundary value problems on a unit square are presented and the analytical solutions are compared with the exact solutions. Both solutions matched perfectly.
Keywords:
Poisson equation, Least squares method,
Grid free method
(19 S., 2001)

26. T. Götz, H. Rave, D. Reinel-Bitzer,
K. Steiner, H. Tiemeier
Simulation of the fiber spinning process

To simulate the influence of process parameters to the melt spinning process a fiber model is used and coupled with CFD calculations of the quench air flow. In the fiber model energy, momentum and mass balance are solved for the polymer mass flow. To calculate the quench air the Lattice Boltzmann method is used. Simulations and experiments for different process parameters and hole configurations are compared and show a good agreement.
Keywords:
Melt spinning, fiber model, Lattice Boltzmann, CFD
(19 S., 2001)

27. A. Zemitis
On interaction of a liquid film with an obstacle

In this paper mathematical models for liquid films generated by impinging jets are discussed. Attention is stressed to the interaction of the liquid film with some obstacle. S. G. Taylor [Proc. R. Soc. London Ser. A 253, 313 (1959)] found that the liquid film generated by impinging jets is very sensitive to properties of the wire which was used as an obstacle. The aim of this presentation is to propose a modification of the Taylor's model, which allows to simulate the film shape in cases, when the angle between jets is different from 180°. Numerical results obtained by discussed models give two different shapes of the liquid film similar as in Taylors experiments. These two shapes depend on the regime: either droplets are produced close to the obstacle or not. The difference between two regimes becomes larger if the angle between jets decreases. Existence of such two regimes can be very essential for some applications of impinging jets, if the generated liquid film can have a contact with obstacles.
Keywords:
impinging jets, liquid film, models, numerical solution, shape
(22 S., 2001)

28. I. Ginzburg, K. Steiner
Free surface lattice-Boltzmann method to model the filling of expanding cavities by Bingham Fluids

The filling process of viscoplastic metal alloys and plastics in expanding cavities is modelled using the lattice Boltzmann method in two and three dimensions. These models combine the regularized Bingham model for viscoplastic with a free-interface algorithm. The latter is based on a modified immiscible lattice Boltzmann model in which one species is the fluid and the other one is considered as vacuum. The boundary conditions at the curved liquid-vacuum interface are met without any geometrical front reconstruction from a first-order Chapman-Enskog expansion. The numerical results obtained with these models are found in good agreement with available theoretical and numerical analysis.
Keywords:
Generalized LBE, free-surface phenomena, interface boundary conditions, filling processes, Bingham viscoplastic model, regularized models
(22 S., 2001)

29. H. Neunzert
»Denn nichts ist für den Menschen als Menschen etwas wert, was er nicht mit Leidenschaft tun kann«
Vortrag anlässlich der Verleihung des Akademiepreises des Landes Rheinland-Pfalz am 21.11.2001

Was macht einen guten Hochschullehrer aus? Auf diese Frage gibt es sicher viele verschiedene, fachbezogene Antworten, aber auch ein paar allgemeine Gesichtspunkte: es bedarf der »Leidenschaft« für die Forschung (Max Weber), aus der dann auch die Begeisterung für die Lehre erwächst. Forschung und Lehre gehören zusammen, um die Wissenschaft als lebendiges Tun vermitteln zu können. Der Vortrag gibt Beispiele dafür, wie in angewandter Mathematik Forschungsaufgaben aus praktischen Alltagsproblemstellungen erwachsen, die in die Lehre auf verschiedenen Stufen (Gymnasium bis Graduiertenkolleg) einfließen; er leitet damit auch zu einem aktuellen Forschungsgebiet, der Mehrskalenanalyse mit ihren vielfältigen Anwendungen in Bildverarbeitung, Materialentwicklung und Strömungsmechanik über, was aber nur kurz gestreift wird. Mathematik erscheint hier als eine moderne Schlüsseltechnologie, die aber auch enge Beziehungen zu den Geistes- und Sozialwissenschaften hat.
Keywords:
Lehre, Forschung, angewandte Mathematik, Mehrskalenanalyse, Strömungsmechanik
(18 S., 2001)

30. J. Kuhnert, S. Tiwari
Finite pointset method based on the projection method for simulations of the incompressible Navier-Stokes equations

A Lagrangian particle scheme is applied to the projection method for the incompressible Navier-Stokes equations. The approximation of spatial derivatives is obtained by the weighted least squares method. The pressure Poisson equation is solved by a local iterative procedure with the help of the least squares method. Numerical tests are performed for two dimensional cases. The Couette flow, Poiseuelle flow, decaying shear flow and the driven cavity flow are presented. The numerical solutions are obtained for stationary as well as instationary cases and are compared with the analytical solutions for channel flows. Finally, the driven cavity in a unit square is considered and the stationary solution obtained from this scheme is compared with that from the finite element method.
Keywords:
Incompressible Navier-Stokes equations, Meshfree method, Projection method, Particle scheme, Least squares approximation
AMS subject classification:
76D05, 76M28
(25 S., 2001)

31. R. Korn, M. Krekel
Optimal Portfolios with Fixed Consumption or Income Streams

We consider some portfolio optimisation problems where either the investor has a desire for an a priori specified consumption stream or/and follows a deterministic pay in scheme while also trying to maximize expected utility from final wealth. We derive explicit closed form solutions for continuous and discrete monetary streams. The mathematical method used is classical stochastic control theory.
Keywords:
Portfolio optimisation, stochastic control, HJB equation, discretisation of control problems.
(23 S., 2002)

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